MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

Vol. 109 No. 6

MEETING OF 1949 NOVEMBER 11

Professor W. M. Smart, President, in the Chair

The election by the Council of the following Fellows was duly confirmed:—

John Richardson Bell, 33 Teesdale Gardens, High Heaton, Newcastle-upon-Tyne, 7 (proposed by F. J. Acfield);

Perry Byerly, Ph.D., Bacon Hall, University of California, Berkeley, California, U.S.A. (proposed by R. Stoneley);

Arthur Bruce Dilks, M.A., 39 Eastern Avenue, Reading, Berkshire (proposed by H. L. Dilks);

Benjamin Hoyle, 34 Gilstead, Bingley, Yorkshire (proposed by F. Hoyle);

Austin Keane, B.Sc., 163 Marrickville Road, Marrickville, New South Wales, Australia (proposed by K. E. Bullen);

Geoffrey Thomas Kneebone, Bedford College, Regent's Park, London, N.W.1 (proposed by C. C. L. Gregory);

Victor Ivor Little, B.Sc., Bedford College, Regent's Park, London, N.W.1 (proposed by C. C. L. Gregory);

Barbara Middlehurst, 4 Clive Crescent, Penarth, Glamorgan (proposed by H. A. Brück);

William Percy Preston, Royal Observatory, Greenwich, London, S.E.10 (proposed by R. d'E. Atkinson); and

Rudolf Tomaschek, Anglo-Iranian Oil Co., Britannic House, Finsbury Circus, London, E.C.2 (proposed by R. d'E. Atkinson).

The election by the Council of the following Junior Members was duly confirmed:—

Joyce Gertrude Gardiner, 38 Ward Avenue, Grays, Essex (proposed by Flora M. McBain);

Josephine Mary Gilloch, Royal Holloway College, Englefield Green, Surrey (proposed by W. H. McCrea);

John Noel Hodgson, 123 Cherrydown Avenue, Chingford, London, E.4 (proposed by R. O. Redman);

Alan Francis Moore, 88 Conyngham Road, Manchester, 14 (proposed by P. M. S. Blackett);

Hans Schneider, M.A., Royal Observatory, Edinburgh, 9 (proposed by M. A. Ellison);

Geoffrey Stephenson, Faircross, 179 High Road, Woodford Wells, Essex (proposed by R. W. B. Pearse); and

Robert Wilson, B.Sc., Royal Observatory, Edinburgh, 9 (proposed by M. A. Ellison).

Ninety-five presents were announced as having been received since the last meeting, including:—

Edgard Vandekerkhove, *Étude des objets célestes resultants* (presented by the author).

MEETING OF 1949 DECEMBER 9

Professor W. M. Smart, President, in the Chair

The President referred to the death on 1949 November 22 of Mr John Henry Reynolds and read the following resolution passed by the Council at its meeting that afternoon (the Fellows standing):

"The Council of the Royal Astronomical Society desires to put on record its deep regret at the loss which the Society has suffered by the death of John Henry Reynolds. His purely scientific work in astronomy was distinguished by his ingenuity in the mechanical design of telescopes and by his observational skill in celestial photography. His munificent gift of a 30-inch reflecting telescope to the Helwân Observatory made possible much early photographic work on the galactic and extra-galactic nebulae; and a second telescope given to the Commonwealth Observatory at Canberra is currently engaged in fresh explorations of the Southern skies. But perhaps his greatest contribution to our science lies in the selfless devotion with which he advanced the aims of the Society by willing service as Treasurer, as Vice-President and as President. On his retirement from office in 1947 he had completed a service totalling more than a quarter of a century, in one capacity or another, on the Council of the Society. By his death British astronomy has lost a devoted servant."

The election by the Council of the following Fellows was duly confirmed:—

Donald Eustace Blackwell, B.A., Pembroke College, Cambridge (proposed by R. O. Redman);

Ronald Newbold Bracewell, B.Sc., M.E., Radiophysics Laboratory, University Grounds, Sydney, New South Wales, Australia (proposed by R. O. Redman);

William Gilmour Guthrie, M.A., Magee University College, Londonderry, Northern Ireland (proposed by E. M. Lindsay);

Daniel L. Harris, Case School of Applied Science, Cleveland, Ohio, U.S.A. (proposed by S. Chandrasekhar);

Daniel Livingston, M.A., 1 Drumbathie Road, Airdrie, Lanarkshire (proposed by W. M. Smart);

John Lowe, A.R.I.C.S., 65 West Regent Street, Glasgow, C.2 (proposed by W. M. Smart);

Malcolm John Miller, 83 Hill Street, Perth, Western Australia (proposed by H. S. Spigl);

Jal Ardeshir Moos, Curzon House, Apollo Bunder, Bombay, India (proposed by B. G. Narayan);

Marcel Nicolet, Institut Royal Météorologique de Belgique, 3 Avenue Circulaire, Uccle, Belgium (proposed by S. Chapman);

Alexander Gregor Robertson, B.Sc., 81 Ormonde Crescent, Muirend, Glasgow, S.4 (proposed by W. M. Smart);

Frank Arthur Roper, M.A., M.D., M.R.C.P., Pendowr, Watergate Bay, Newquay, Cornwall (proposed by W. M. Lindley);

Robert Keith Simpson, 8 Kingsford Avenue, Muirend, Glasgow, S.4 (proposed by W. M. Smart);

Michael Jessop Smyth, B.A., Solar Physics Observatory, Cambridge (proposed by R. O. Redman);

Thomas Russell Tannahill, M.A., B.Sc., Ph.D., 98 Speirs Road, Bearsden, Glasgow (proposed by W. M. Smart);

Leslie Hamilton Tarrant, Anglo-Iranian Oil Co., Kirklington Hall, nr. Newark, Nottinghamshire (proposed by B. C. Browne);

Elizabeth Isobelle Thomson, Arley Castle, nr. Bewdley, Worcestershire (proposed by W. M. Smart);

Roy Henry Tucker, Royal Observatory (Greenwich), Abinger Common, nr. Dorking, Surrey (proposed by H. Spencer Jones);

Edgard Vandekerkhove, D.Sc., Observatoire Royal de Belgique, Uccle, Belgium (proposed by P. J. Melotte); and

Gérard de Vaucouleurs, D.Sc., Institut d'Astrophysique, 98 bis Boulevard Arago, Paris, 14, France (proposed by E. M. Burbidge).

The election by the Council of the following Junior Members was duly confirmed:—

Donald George Ewart, 38 Osbourne Avenue, Jesmond, Newcastle-upon-Tyne (proposed by W. M. Smart);

David George Randall, 41 Cambridge Road, Marlow, Buckinghamshire (proposed by G. J. Whitrow); and

Frederick Steel, 27 Sydney Grove, Hendon, London, N.W.4 (proposed by C. C. L. Gregory).

Eighty-one presents were announced as having been received since the last meeting, including:—

Atmospheric pressure and temperature measurements between the altitudes of 40 *and* 110 *kilometers* (presented by the Department of Electrical Engineering Research Institute of the University of Michigan); and

T. J. J. See, *Wave Theory Discovery of the Cause of Gravitation* (presented by the author).

Dr R. W. B. Pearse and Dr J. C. P. Miller were appointed honorary auditors of the Treasurer's accounts for the year 1949.

THE STRUCTURE OF STARS OF NON-UNIFORM COMPOSITION

F. Hoyle and R. A. Lyttleton

(Received 1949 May 13)

Summary

It is shown that stellar models of non-uniform composition can be constructed on the basis of standard theory to give the largest distensions of radius that are indicated by observations of red giants. A selection of the results obtainable by the methods here developed are plotted in Fig. 1, which also shows the main-sequence curve, the giant sequence for populations of Type I, and the giant sequence for populations of Type II. The possibility of selection effects in relation to this diagram is discussed.

Consideration of the physical factors that determine whether a star shall be of uniform or non-uniform composition shows why in the case of Type I population the giant sequence is separated by a gap from the main sequence, whereas in Type II population it is connected directly with the main sequence.

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1. *Introduction.*—This paper is an extension of previous work * in which it was shown, on the basis of certain approximations, that a discontinuous decrease of mean molecular weight in the atmosphere of a star may lead to a substantial increase of its radius. The approximations made were sufficiently good to enable it to be shown by means of integrations that an increase by a factor of about three could certainly occur, thereby demonstrating the potentialities of the hypothesis, and further calculations strongly suggested the possibility of far greater increases. But the discussion of this given in the earlier work nevertheless fell short of explaining with certainty the distensions of the largest known red giants, which require increases by a factor of about 100, as compared with the radii of ordinary main-sequence stars. The chief difficulty with this part of the work was the neglect of the mass of the atmosphere, which for such large extensions could not be sufficiently accurately allowed for by the method used. The re-examination of this question to a higher degree of approximation is the chief object of the present investigation. The discussion shows that the largest distensions as yet required by observation can be obtained theoretically.

It will be convenient to regard the star as possessing an inner region with constant molecular weight μ, which itself consists of a central convective core, whose radius will be denoted by r_c, and a radiative region whose exterior radius will be denoted by r_1. Surrounding this it will be supposed there is an outer region in which the molecular weight is $\frac{1}{2}$, corresponding to material composed effectively solely of hydrogen. The suffixes c and 1 will also be used to indicate the particular values of the various physical quantities at the surface of the core and at the interface where the molecular weight changes. Thus $\boldsymbol{T}_c$ will denote the temperature at the edge of the convective core, and $\boldsymbol{T}_1$ the temperature at the interface.

* F. Hoyle and R. A. Lyttleton, *M.N.*, **102**, 218, 1942.

In our earlier investigation, the generation of energy in the star was assumed to take place in accordance with the Carbon–Nitrogen cycle, so that in the central regions density and temperature distributions were closely similar to those of main-sequence stars. The same assumption and consequences will also hold in the present investigation.

The physical importance and likely result of a discontinuous decrease of mean molecular weight are easily understood in general terms, for the effect is to produce a discontinuous increase in the ratio of the thermal energy per unit mass of the material to its gravitational energy per unit mass. If this thermal energy exceeded the gravitational energy the outer region might be expected to extend to infinity, but it is found in all the examples studied that this does not actually occur because of the additional gravitational field resulting from the mass of the outer region itself. The radius nevertheless becomes very large in several of the cases considered.

So far no one has succeeded in obtaining stellar models wholly free from both mathematical and physical approximations. Some form of simplification has always had to be introduced, and the decision that invariably has to be made is whether to maintain a closely correct physical picture and make mathematical approximations to deal with the resulting equations, or to make considerable simplifications of the physical picture and thereafter maintain as strict a mathematical treatment as possible. While experience shows that, from the point of view of obtaining physical insight into the matters concerned, the first alternative is much to be preferred, it is something of a paradox that many investigators have followed the second plan. This may be due to the greater esteem in which work of the latter kind appears to be held, its conclusions often being considered, incorrectly, to have something of the certainty of the conclusions of pure mathematics, possibly because of the resemblance of the methods of treatment. But it is important to realize that the conclusions of such work cannot have greater weight than the initial assumptions on which it is based, and if these are oversimplified the conclusions may be too rarefied to be of much value. It is the view of the writers that in most, if not all, physical theory where approximation is necessary the first alternative is generally to be preferred, and that this is particularly so in the theory of stellar structure.* But there is an ever-present difficulty in adopting the first method, and this is that only considerable familiarity with the problem shows what mathematical approximations can safely be made. Thus, it has taken about thirty years for general confidence to be felt about what approximations are valid for the theory of stars of uniform composition. As a result it is now known, for example, that it is not necessary to make Eddington's original physical assumption that ηk is constant in order that β, the radiation-pressure parameter, may be assumed constant throughout the star. It is a permissible approximation to put β constant whatever the model. It is also known that, so long as the physical conditions at r_c and the surface of the star are closely satisfied, the form of the opacity law has little effect on the computed radius. On the other hand, for an adequate calculation of the luminosity, a detailed treatment of the opacity is essential. Again, it is now known that the theory is

* See, for instance, A. S. Eddington, *Internal Constitution of the Stars*, Cambridge, p. 101, 1930. Eddington himself evidently experienced criticism on this score, and here uses his full literary powers in saying why the procedure of applied mathematics is to be preferred.

extremely insensitive to the cross-sections for the nuclear processes that are responsible for energy-generation in the stars.

When we turn from stars of uniform composition to the red giants, the question of what approximations will now be admissible has to be considered afresh. The first and simplest possibility that suggested itself was that it might be permissible to neglect the mass of the outer extended region. As explained above, however, this is satisfactory for moderate increase of radius, but is unduly restrictive when large extensions are involved. Further study of the problem shows that two approximations of a less restrictive nature can be satisfactorily made that enable the work to be suitably extended to deal with models of very large radius. The approximations are found to simplify not only the arithmetical calculations, but also the presentation of the work. The nature of these approximations will next be explained.

(*a*) In calculating the mass distribution in the outer region the effective polytropic index may be assumed constant. The temperature distribution, and hence the radius, can be found to high approximation by making use of this calculated mass distribution.

The validity of this approximation may be understood by considering for example the case of opacity by electron scattering. When this holds, the effective index n is determined by a relation of the form

$$n+1=AMT^3/\rho,$$

where A is a certain constant, T and ρ are the temperature and density at the central distance r at which n is being computed, and M is the mass interior to this distance. Now it is known that n increases outwards from a value $\geqslant 1{\cdot}5$ at $r=r_1$ to a value of 3 at the surface. Thus, the maximum permissible variation of $n+1$ in the outer region is small compared with the variations of T and ρ, which quantities change by large factors. The change of M for the outer region is also found to be comparatively small in the cases of physical interest, and it follows therefore that even if it were calculated accurately ρ would vary almost as T^3 in this region. But $(n+1)$ is also given by $d\log(\rho T)/d\log T$, so that it must in fact be close to 3 everywhere in the outer region. As a general rule it may be noted that n can only change appreciably in a region for which the fractional change of mass $\delta M/M$ is comparable with, or greater than, $\delta T/T$ and $\delta\rho/\rho$.

It will be realized that it is not simply being assumed that the distribution in the outer region is $\rho \propto T^3$, which would be equivalent to ignoring the contribution to the mass by this region. What is being done is that in certain terms whose influence is small this approximation is used. This step is in accordance with the standard procedure of successive approximation in applied mathematics where the effect of terms of a given order can be evaluated by finding the disturbing influence with values corresponding to the next lower order.

(*b*) Radiation pressure can be neglected. There are two reasons for this. First, the effect of radiation pressure on the radius of a star is small (in contrast with the effect on the luminosity, which may be important) and second, the value of the parameter β is always near unity for the models of chief interest.

It may be useful to mention at the outset those features on which the radius of a red giant is found to depend strongly. They are:—

(i) the value of μ in the inner region (the molecular weight in the outer region is assumed throughout to be $\frac{1}{2}$);

(ii) the ratio of the mass in the outer region to that of the inner region;
(iii) the form of the opacity law in the outer region.

Where the opacity is concerned, present-day indications of the chemical composition of stars suggest that apart from hydrogen and helium the only elements needing consideration are carbon, nitrogen and oxygen, which together comprise as a rule a few per cent of the total mass. (For the Sun they amount to about 3 per cent.) In the central regions of a star the radiation quanta have energies far higher than the absorption edges of the K-shells of these elements, and for this reason their contribution to the opacity is negligible. But in the cases of interest, at $r = r_1$ the temperature is only about one-fifth the central value, and the contribution of the K-electrons of these elements becomes important if their concentrations are sufficiently high. It can be estimated, by standard methods, that a concentration of about 10 per cent by mass is necessary for the C–N–O group to control the opacity in the outer region. Accordingly, in order to bracket the whole range of possible cases we have considered two models:—

(I) *The electron-scattering model.*—This corresponds to the case for which the concentration of the C–N–O group is so small that the contribution to the total opacity is negligible. The opacity is then due to electron scattering and the polytropic index may be taken as 3 everywhere in the outer region.

(II) *The photoelectric model.*—In this the concentration of the C–N–O group is regarded as sufficiently high for its opacity to be the controlling influence in the outer region.

It is convenient mathematically, however, to regard electron scattering as being the chief source of opacity throughout the inner region. Accordingly, in this model there is not only a discontinuous change of molecular weight at $r = r_1$, but also in the form of the opacity law. To allow for the guillotine factor the polytropic index is taken as 4 in the whole of the outer region instead of the value 3·25 that would follow simply from Kramers' law. Such a procedure tends to exaggerate the effect of the guillotine factor, but this has the advantage in the present investigation that all physical cases are more certain to lie somewhere between the two extreme models adopted.

2. *The photoelectric model.*—We adopt the following notation:—

$\bar{\rho}$ = density at distance r in the inner region.
$\bar{T}$ = temperature at distance r in the inner region.
$\bar{\rho}$ = density at distance r in outer region.
$\bar{T}$ = temperature at distance r in outer region.
$\bar{M}(r)$ = mass in grams within distance r.
X = the (constant) proportion by mass of hydrogen in the inner region.
L = the luminosity of the star.
$\mathfrak{R}$ = the gas constant.
G = the constant of gravitation.
a = Stefan's constant.
c = the speed of light.

The suffix c indicates values at $r = r_c$, and the suffix 1 at $r = r_1$. C.g.s. units are used for all the above quantities.

The equations governing this model are:—

$$\frac{d}{dr}(\bar{\rho}\overline{T}) = -\frac{G\mu}{\mathfrak{R}}\cdot\frac{\bar{\rho}\overline{M}}{r^2}, \qquad r<r_1, \tag{1}$$

$$\frac{d}{dr}(\bar{\rho}\overline{T}) = -\frac{G}{2\mathfrak{R}}\cdot\frac{\bar{\rho}\overline{M}}{r^2}, \qquad r>r_1, \tag{2}$$

$$\frac{d\overline{M}}{dr} = 4\pi r^2\bar{\rho}, \qquad r<r_1, \tag{3}$$

$$\frac{d\overline{M}}{dr} = 4\pi r^2\bar{\rho}, \qquad r>r_1, \tag{4}$$

$$\bar{\rho} \propto \overline{T}^{3/2}, \qquad r\leqslant r_c, \tag{5}$$

$$\frac{d\overline{T}}{dr} = -\frac{0{\cdot}57(1+X)\bar{\rho}L}{16\pi ac\overline{T}^3r^2}, \qquad r_c<r<r_1, \tag{6}$$

$$\frac{d\overline{T}}{dr} = -\frac{0{\cdot}57\overline{T}_1\bar{\rho}L}{8\pi ac\overline{T}^4r^2}, \qquad r>r_1, \tag{7}$$

$$\overline{T}_1=\overline{T}_1 \qquad \text{and} \qquad \bar{\rho}_1=2\mu\bar{\rho}_1. \tag{8}$$

The conditions (8) obtaining at r_1 express the continuity of the temperature and pressure across the interface at which μ changes.

The assumed difference of composition between the inner and outer regions may be regarded as arising physically from the conversion of hydrogen to helium within the inner region. It is reasonable therefore to suppose that the concentration by mass of the C–N–O group of elements will be the same on both sides of the interface, and it is then a necessary physical condition that the opacity per unit mass shall change discontinuously at r_1 in the same way as the electron density. The numerical coefficient on the right-hand side of (7) has accordingly been chosen so that this condition is satisfied.

It will be noticed that (7) is not the only possible equation of transfer leading to polytropic index 4 in the outer region, for any equation of the form

$$\frac{d\overline{T}}{dr} = -\frac{0{\cdot}57LT_1^{4q-3}}{8\pi acr^2\bar{\rho}_1^{q-1}}\left(\frac{\bar{\rho}}{\overline{T}^4}\right)^q \qquad q>0$$

also leads to $\bar{\rho}$ varying approximately as $\overline{T}^4$, and therefore to the polytropic index n being close to 4. A method similar to that actually given below shows that all such equations lead to identical results, as expressed in the final equation (23) that determines the variation of $\overline{T}$ with radial distance, and there is therefore no loss of generality in adopting (7).

For numerical purposes it is more convenient to express the density and temperature everywhere as proportions of their central values $\bar{\rho}_0$, $\overline{T}_0$ (say), and when so expressed they will be denoted by ρ, T in the inner region, $r<r_1$, and ρ, T in the outer region $r>r_1$. The mass $M(r)$ is expressed in dimensionless form as a multiple of the quantity

$$4\pi\bar{\rho}_0\{5\mathfrak{R}\overline{T}_0/8\pi\mu G\bar{\rho}_0\}^{3/2}$$

and is then denoted by M. Also r is replaced by the dimensionless variable z related to it by

$$r = \{5\mathfrak{R}\bar{T}_0/8\pi\mu G\bar{\rho}_0\}^{1/2} z.$$

In terms of these variables the equations (1)–(8) become

$$\frac{d}{dz}(\rho T) = -2{\cdot}5\rho M/z^2, \qquad z<z_1, \tag{9}$$

$$\frac{d}{dz}(\rho T) = -\frac{1{\cdot}25\rho M}{\mu z^2}, \qquad z>z_1, \tag{10}$$

$$\frac{dM}{dz} = \rho z^2, \qquad z<z_1, \tag{11}$$

$$\frac{dM}{dz} = \rho z^2, \qquad z>z_1, \tag{12}$$

$$\rho = T^{3/2}, \qquad z\leqslant z_c, \tag{13}$$

$$\frac{dT}{dz} = -\frac{\kappa\rho}{z^2T^3}, \qquad z_c\leqslant z<z_1, \tag{14}$$

$$\frac{dT}{dz} = -\frac{2\kappa T_1\rho}{(1+X)z^2T^4}, \qquad z>z_1, \tag{15}$$

$$T_1 = T_1, \qquad \rho_1 = 2\mu\rho_1. \tag{16}$$

In equations (14) and (15) the factor κ is proportional to L and must be such as to make the value of the temperature gradient at the beginning of the inner radiative zone continuous with that at the outer limit of the convective core in order that the energy flux shall be continuous across its boundary.

The effective polytropic index is defined as usual by

$$n+1 = \frac{d\log(\rho T)}{d\log T} = \frac{2{\cdot}5MT^3}{\kappa\rho}, \qquad z<z_1, \tag{17}$$

$$n+1 = \frac{d\log(\rho T)}{d\log T} = \frac{1{\cdot}25(1+X)MT^4}{2\kappa\mu T_1\rho}, \qquad z>z_1. \tag{18}$$

Since the form of (15) requires ρ to vary almost as T^4 when $z>z_1$, it follows that n cannot differ much from 4, and equation (18) gives, to close approximation,

$$8 = \frac{(1+X)MT^4}{\kappa\mu T_1\rho}. \tag{19}$$

From (15) and (19) we now obtain

$$z^2\frac{dT}{dz} = -\frac{M}{4\mu}. \tag{20}$$

This may be put in a form more convenient for calculation by writing

$$z = z_1x, \qquad T = T_1\tau,$$

when (20) becomes

$$T_1z_1x^2\frac{d\tau}{dx} = -\frac{M}{4\mu}. \tag{21}$$

We also have from (12)

$$M = M_1 + \int_{z_1}^{z} \rho z^2 \, dz,$$

which in terms of x becomes

$$\frac{M(x)}{M_1} = 1 + \frac{\rho_1 z_1^3}{M_1} \int_1^x x^2 \tau^4 \, dx, \tag{22}$$

since to sufficient accuracy we can here put $\rho = \rho_1 \tau^4$, this term itself being small in the cases of interest found later. Thus the equation determining the variation of temperature in the outer region takes the form

$$x^2 \frac{d\tau}{dx} = -\xi - \zeta \int_1^x x^2 \tau^4 \, dx, \tag{23}$$

where ξ and ζ are certain constants defined by

$$\xi = \frac{M_1}{4\mu z_1 T_1}, \qquad \zeta = \frac{\xi \rho_1 z_1^3}{M_1}. \tag{24}$$

Before proceeding to solve (23) it is desirable to consider what values the numerical factors ξ and ζ are likely to have. For this purpose, if we apply (19) at z_1 and (17) at z_c, we obtain, after putting $n_c = 1{\cdot}5$, by eliminating κ,

$$\frac{\rho_1 z_1^3}{M_1} = (z_1 T_1)^3 \frac{1+X}{8\mu} \cdot \frac{\rho_c}{M_c T_c^3}. \tag{25}$$

Thus ζ may be obtained by multiplying the right-hand side of (25) by ξ, when ξ is settled.

The value of X may be assumed given, and hence the factor $(1+X)/8\mu$ may be regarded as known, since

$$\mu = \frac{4}{3+5X},$$

the small proportion of heavy elements being neglected so that $X + Y \doteqdot 1$, where Y = proportion by mass of helium.

Where the choice of a parameter to describe the possible series of configurations is concerned, the natural physical quantity to use (once X is given) would be the ratio of the total mass in the outer region to that of the inner region, but mathematically this would be extremely awkward. There is however a one–one correspondence between this quantity and z_1, so that z_1 itself could be used. The temperature T_1 is also in one–one correspondence with z_1, and it has been found that so also is the product $z_1 T_1$, which is an equivalent parameter, and this turns out more convenient than any of these other possibilities. When $z_1 T_1$ is specified, z_1, z_c and the whole solution for $z > z_c$ are also determined, as will be seen, and we shall accordingly adopt $z_1 T_1$ as the parameter in the subsequent work.

This settled, we return to the question of the numerical values of ξ and ζ, as given by (23) and (25). The unknown quantities in these expressions may be regarded as M_1 and $\rho_c / M_c T_c^3$. Now experience of a large number of numerical integrations has shown that the core radius z_c is extraordinarily insensitive to the value selected for z_1 or $z_1 T_1$, and that ample accuracy for both M_1 and $\rho_c / M_c T_c^3$ can be achieved by retaining the same fixed value of z_c for all $z_1 T_1$. This procedure, at the same time as being highly accurate, saves a great deal of tedious

and profitless arithmetic. In the following work we use values corresponding to the case $z_c = 1{\cdot}58$ computed by C. M. and H. Bondi in their recent paper.* The solution gives $T_c = 0{\cdot}652$, $\rho_c = 0{\cdot}526$, $M_c = 0{\cdot}910$ and leads immediately to the values given in the following table:—

TABLE I

$z_1 T_1$	0·3	0·4	0·5	0·6	0·7	0·8	0·9
z_1	4·54	4·20	3·89	3·59	3·31	2·97	2·62
M_1	2·75	2·74	2·73	2·71	2·62	2·48	2·29
$\mu\xi$	2·29	1·71	1·37	1·13	0·936	0·775	0·636
$\frac{\mu^2\zeta}{1+X}$	0·0161	0·0286	0·0445	0·0636	0·0837	0·1035	0·1209

These values lead to the following numerical values for ξ and ζ as given by Tables II and III, respectively.

TABLE II

$z_1 T_1$	0·3	0·4	0·5	0·6	0·7	0·8	0·9
$\mu = \frac{2}{3}$	3·44	2·57	2·05	1·69	1·40	1·16	0·954
$\mu = 1$	2·29	1·71	1·37	1·13	0·936	0·775	0·636
$\mu = \frac{4}{3}$	1·72	1·28	1·02	0·847	0·702	0·581	0·477

TABLE III

$z_1 T_1$	0·3	0·4	0·5	0·6	0·7	0·8	0·9
$\mu = \frac{2}{3}$	0·0581	0·103	0·160	0·229	0·301	0·373	0·435
$\mu = 1$	0·0194	0·0343	0·0534	0·0763	0·1004	0·1242	0·1451
$\mu = \frac{4}{3}$	0·0091	0·0161	0·0250	0·0358	0·0471	0·0582	0·0680

It will be seen from the foregoing table that in most cases ζ is small and it will become clear later that in the cases of most importance it is always near nought. A zero-order approximation to (23) is therefore obtained by neglecting the integral on the right, when the solution is easily found to be

$$\tau = \xi/x - \xi + 1. \tag{26}$$

A first-order approximation for τ could be found by substituting (26) in the mass term in (23) and solving the resulting equation, but this procedure is unnecessarily elaborate, as can be seen at this stage from the following argument, and as is also borne out later by the detailed calculations. At the surface of the star $\tau = 0$, and hence $x_s = \xi/(\xi - 1)$. Accordingly ξ must be close to unity in models giving large x_s. Thus in such models, which from the point of view of the present paper are the ones of chief interest, it is sufficient to replace (26) by $\tau = 1/x$ as giving a valid zero-order approximation to be used in the mass term in (23). The relation $\tau = 1/x$ does not give so close an approximation in cases where ξ is appreciably greater than unity, but it may nevertheless be used in the mass term in (23), since this term is itself small in these cases. In short, $\tau = 1/x$ can be used as the zero-order approximation in all cases.

To obtain the next approximation we accordingly put $\tau = 1/x$ on the right-hand side of (23), and the equation then integrates at once to give

$$\tau = \frac{\xi + \zeta}{x} - \frac{\zeta}{2x^2} - \frac{\zeta}{2} - \xi + 1. \tag{27}$$

* C. M. and H. Bondi, *M.N.*, **109**, 62–85, 1949. (The above values have been derived by interpolation from Table I, p. 76.)

The surface value of x, which we denote by x_s, is found by putting $\tau = 0$. Since in the cases of importance $\zeta \ll 1$ and x_s is large, we can get an estimate of x_s at this stage by neglecting the term $\zeta/2x^2$, and then $\tau = 0$ leads to

$$x_s \doteq (\xi + \zeta)/(\xi + \tfrac{1}{2}\zeta - 1), \tag{28}$$

and this shows that the present approximation will not be adequate to deal with cases for which $\xi + \frac{1}{2}\zeta \leqslant 1$. Such cases demand the use of the second approximation considered below, but before proceeding to this it will be useful to tabulate values of x_s for cases in which the present approximation is adequate, as despite their approximate nature they serve to give an indication of the cases likely to be most important.

TABLE IV

$z_1 T_1$	0·3	0·4	0·5	0·6	0·7	0·8	0·9
$\mu = 2/3$	1·42	1·65	1·96	2·38	3·07	4·40	8·10
$\mu = 1$	1·78	2·39	3·62	7·21	...	...	...
$\mu = 4/3$	2·39	4·43	...	...	...	...	...

The vacant places in this table are those corresponding to cases for which $\xi + \frac{1}{2}\zeta < 1$.

We next calculate for various values of $z_1 T_1$ the ratio of the mass in the outer region to that of the inner region. To the present approximation this, from (22), is given by

$$\frac{M - M_1}{M_1} = \frac{\zeta}{\xi}\left(1 - \frac{1}{x_s}\right) = \frac{\zeta(1 + \frac{1}{2}\zeta)}{\xi(\xi + \zeta)}. \tag{29}$$

Values of this ratio, for $\mu = \frac{2}{3}$, 1 and $\frac{4}{3}$, for a range of values of $z_1 T_1$ are given in the following table :—

TABLE V

Ratio of the mass in outer region to mass of inner region

$z_1 T_1$	0·3	0·4	0·5	0·6	0·7	0·8	0·9
$\mu = \frac{2}{3}$	0·005	0·016	0·028	0·078	0·145	0·248	0·400
$\mu = 1$	0·004	0·012	0·028	0·058	...	...	...
$\mu = \frac{4}{3}$	0·003	0·010	...	...	...	...	...

The second approximation.—We now consider the higher approximation to the solution for the particular case $\xi + \frac{1}{2}\zeta = 1$. Equation (27) reduces to

$$\tau = \frac{1 + \frac{1}{2}\zeta}{x} - \frac{\zeta}{2x^2}, \tag{30}$$

and the second approximation is obtained by putting this expression for τ in the integral on the right-hand side of (23). We find

$$\int_1^x x^2 \tau^4 \, dx = \frac{2}{5\zeta}\left\{\left(1 + \frac{\zeta}{2} - \frac{\zeta}{2x}\right)^5 - 1\right\}, \tag{31}$$

and therefore (23) becomes

$$x^2 \frac{d\tau}{dx} = (\tfrac{1}{2}\zeta - 1) - \frac{2}{5}\left\{\left(1 + \frac{\zeta}{2} - \frac{\zeta}{2x}\right)^5 - 1\right\}. \tag{32}$$

Taking account of the condition $\tau = 1$ at $x = 1$, the integral is

$$\tau = \frac{\frac{3}{5} - \frac{1}{2}\zeta}{x} + \tfrac{1}{2}\zeta + \frac{2}{5} - \frac{2}{15\zeta}\left\{\left(1 + \frac{\zeta}{2} - \frac{\zeta}{2x}\right)^6 - 1\right\}. \tag{33}$$

When x is large, this expression for τ can be satisfactorily approximated to by

$$\tau = \frac{1}{x}(1+\tfrac{1}{2}\zeta)+\tfrac{1}{2}\zeta+\frac{2}{5}-\frac{2}{15\zeta}\{(1+\tfrac{1}{2}\zeta)^6-1\}, \tag{34}$$

and $\tau = 0$ gives for the radius

$$x_s = (1+\tfrac{1}{2}\zeta)\left\{\frac{2}{15\zeta}[(1+\tfrac{1}{2}\zeta)^6-1]-\tfrac{1}{2}\zeta-\frac{2}{5}\right\}^{-1}$$

$$\doteqdot \frac{3}{\zeta^2}(1+\tfrac{1}{8}\zeta-\tfrac{1}{8}\zeta^2). \tag{35}$$

According to the tables given above, for the case $\mu = 1$, $\xi+\frac{1}{2}\zeta = 1$ when $\zeta = 0{\cdot}098$ about. Substituting in (35) gives $x_s \doteqdot 315$. Similarly for $\mu = \frac{4}{3}$, $\xi+\frac{1}{2}\zeta = 1$ when $\zeta = 0{\cdot}027$ about, and (35) gives $x_s \doteqdot 4130$. (For $\mu = \frac{2}{3}$, $x_s \doteqdot 12{\cdot}4$.)

Remembering that for main-sequence stars the value of x_s is close to 2, it is evident that the foregoing models have greatly extended radii. Moreover, further approximation on the present theory would not reduce these values of x_s, for the expression (30) gives, for large x, an overestimate for τ, and consequently the temperature gradient is overestimated by using (30) in the right-hand side of (23). This in turn means that (34) gives an underestimate for τ, and therefore that (35) is an underestimate for x_s.

Nevertheless, for physical reasons not covered by the present theory, these results must be subject to reduction. To see why this is, we have, especially near the surface, that

$$T(x) \doteqdot \left(\frac{1}{x}-\frac{1}{x_s}\right)T_1, \qquad \rho(x) \doteqdot \left(\frac{1}{x}-\frac{1}{x_s}\right)^4 \rho_1. \tag{36}$$

Now in cases of physical interest ρ_1 is of the order of 1 g. cm.$^{-3}$ and T_1 is of the order of 10^7 deg. K. Thus for $x = \frac{1}{2}x_s$ relations (36) imply

$$T(\tfrac{1}{2}x_s) \sim 10^7 x_s^{-1}, \qquad \rho(\tfrac{1}{2}x_s) \sim x_s^{-4}.$$

For $x_s = 315$ these give $T \sim 30{,}000$ deg. K. and $\rho \sim 10^{-10}$ g. cm.$^{-3}$; while for $x_s = 4130$ they give $T \sim 2500$ deg. K. and $\rho \sim 3 \times 10^{-14}$ g. cm.$^{-3}$. But the opacity formula on which the whole calculations are based cannot be extrapolated to such low values of the density as these. According to studies of stellar atmospheres, the opacity must in any actual case cease to be adequate to confine the radiation when ρ drops below a value of order 10^{-9} g. cm.$^{-3}$. If in the present cases the level at which ρ falls to this is taken as defining an effective radius for the star, the results require a value $x_s \sim 100$ in the case $\mu = 1$; and a value $x_s \sim 200$ for $\mu = \frac{4}{3}$. Further consideration of these results will be deferred to Section 4.

The proportion of mass in the outer region can readily be found by means of (22) and (31), which show that the ratio of the mass of the outer region to that of the inner region is

$$\frac{M-M_1}{M_1} = \frac{2}{5\xi}\left\{\left(1+\frac{\zeta}{2}-\frac{\zeta}{2x_s}\right)^5-1\right\}. \tag{37}$$

For the foregoing numerical case, viz. $\zeta = 0{\cdot}098$, $x_s = 315$, this ratio is about 0·088 for $\mu = 1$, and about 0·022 for $\mu = \frac{4}{3}$. It is remarkable that such large extensions of radius are accompanied by only these moderate proportions of mass in the outer region.

There is the further question of what happens if the mass ratio should exceed substantially a value of the above order. The model in such cases is characterized by the value of $\xi+\frac{1}{2}\zeta$ falling below unity, and its properties can be sufficiently illustrated by considering a numerical example. Suppose we take $\xi=0{\cdot}775$, $\zeta=0{\cdot}124$. Then, by (27), the first approximation for τ is

$$\tau=\frac{0{\cdot}899}{x}-\frac{0{\cdot}062}{x^2}+0{\cdot}163, \tag{38}$$

and inserting this in (23) the second approximation is found to be

$$\begin{aligned}\tau=\frac{0{\cdot}885}{x}-\frac{0{\cdot}040}{x^2}&+0{\cdot}153-0{\cdot}0000146x^2-0{\cdot}0009934x\\&-0{\cdot}01585\log x+0{\cdot}05657\frac{\log x}{x}\\&+\text{negligible terms in } x^{-3},\ x^{-4},\ x^{-5} \text{ and } x^{-6}.\end{aligned} \tag{39}$$

The solution of this for $\tau=0$ is found by numerical methods to be approximately $x_s=60$.

The proportion of mass in the outer region can be found by means of (22) and has been calculated by numerical integration with values of τ obtained from (39). This leads to $(M-M_1)/M_1=2{\cdot}3$, which is much larger than the former values corresponding to $\xi+\frac{1}{2}\zeta=1$. But as compared with the case $\xi+\frac{1}{2}\zeta=1$, this large excess mass makes no substantial change in either the radius of the star or its luminosity, which may be computed by considering the inner region as if it were the inner region of a main-sequence star. Accordingly the excess mass would have no important effect on the plotted position of the star in a luminosity-type diagram. On the other hand, such a star would appear under-luminous if plotted in a mass-luminosity diagram, and would lie on the appropriate μ-curve only if the mass of its inner region were used in place of the total mass.

3. *The electron-scattering model.*—Calculations corresponding to those of Section 2 will now be carried out for the second model. The relevant equations are indicated by the same numbers as before but with an accent to denote the form appropriate to the present model. Thus in place of (6) we how have

$$\frac{d\bar{T}}{dr}=-\frac{0{\cdot}57\bar{\rho}L}{8\pi acr^2\bar{T}^3}, \qquad r>r_1. \tag{6'}$$

Similarly, corresponding to (15), (19), (20), (23) and (25), we have

$$\frac{dT}{dz}=-\frac{2\kappa\rho}{(1+X)z^2T^3}, \qquad z>z_1. \tag{15'}$$

$$8=1{\cdot}25(1+X)MT^3/\kappa\mu\rho. \tag{19'}$$

$$z^2\frac{dT}{dz}=-2{\cdot}5M/8\mu. \tag{20'}$$

$$x^2\frac{d\tau}{dx}=-\xi-\zeta\int_1^x x^2\tau^3\,dx, \tag{23'}$$

where now

$$\xi=\frac{2{\cdot}5M_1}{8\mu z_1T_1} \qquad \text{and} \qquad \zeta=\frac{\rho_1z_1^3}{M_1}\xi,$$

and finally

$$\frac{\rho_1z_1^3}{M_1}=(z_1T_1)^3\frac{1{\cdot}25(1+X)}{8\mu}\cdot\frac{\rho_c}{M_cT_c^3}. \tag{25'}$$

It is seen that the values previously obtained for ξ and ζ, as functions of z_1T_1, if multipled by $\frac{5}{4}$ and $\frac{25}{16}$ respectively, become the values appropriate to the present case. We therefore have the following sets of values of ξ and ζ, respectively:—

TABLE VI

z_1T_1	0·3	0·4	0·5	0·6	0·7	0·8	0·9
$\mu=\frac{2}{3}$	4·30	3·21	2·56	2·12	1·76	1·45	1·19
$\mu=1$	2·87	2·14	1·71	1·41	1·17	0·969	0·795
$\mu=\frac{4}{3}$	2·15	1·61	1·28	1·06	0·877	0·726	0·596

TABLE VII

z_1T_1	0·3	0·4	0·5	0·6	0·7	0·8	0·9
$\mu=\frac{2}{3}$	0·0907	0·161	0·250	0·358	0·471	0·582	0·680
$\mu=1$	0·0303	0·0536	0·0834	0·119	0·157	0·194	0·227
$\mu=\frac{4}{3}$	0·0142	0·0251	0·0391	0·0559	0·0735	0·0910	0·106

For the integration of (23′) we again adopt a method of successive approximation, putting $\tau=1/x$ in the integral on the right in order to obtain the first-order approximation. This leads to

$$\tau=\frac{1}{x}\{\xi+\zeta(1+\log x)\}+(1-\xi-\zeta). \qquad (27')$$

So long as $(1-\xi-\zeta)$ is negative and not too small, the value of x_s is given by putting the right-hand side of (27′) equal to zero. In these cases, the term $\log x$ can be neglected with sufficient accuracy, so that

$$x_s=(\xi+\zeta)/(\xi+\zeta-1).$$

The neglect of the $\log x$ term leads to x_s being underestimated.

The ratio of the mass in the outer region to the mass in the inner region is now given by $(\zeta\log x_s)/\xi$.

The following values have been computed by means of this first approximation. (The spaces in the last line in each table correspond to values of z_1T_1 for which $\xi+\zeta$ is negative.)

TABLE VIII

Table for x_s

z_1T_1	0·3	0·4	0·5	0·6	0·7	0·8	0·9
$\mu=\frac{2}{3}$	1·30	1·42	1·55	1·68	1·82	1·97	2·15
$\mu=1$	1·53	1·84	2·27	2·89	4·06	7·14	46·9
$\mu=\frac{4}{3}$	1·86	2·59	4·13	9·70	...	...	...

TABLE IX

Ratio of the mass in outer region to mass of inner region

z_1T_1	0·3	0·4	0·5	0·6	0·7	0·8	0·9
$\mu=\frac{2}{3}$	0·006	0·018	0·043	0·088	0·160	0·271	0·435
$\mu=1$	0·004	0·015	0·040	0·089	0·188	0·394	1·10
$\mu=\frac{4}{3}$	0·004	0·015	0·043	0·119	...	...	...

We next consider the special case $\xi+\zeta=1$. For this (27′) becomes

$$\tau=(1+\zeta\log x)/x. \qquad (30')$$

We also have, using this value of τ,

$$\int_1^x x^2\tau^3\,dx=\{(1+\zeta\log x)^4-1\}/4\zeta. \qquad (31')$$

Thus (23′) becomes

$$x^2\frac{d\tau}{dx}-\zeta-1-\tfrac{1}{4}\{(1+\zeta\log x)^4-1\}, \qquad (32')$$

which admits of simple integration to

$$\tau=\frac{1}{x}\{(\tfrac{3}{4}-\zeta+6\zeta^4)+6\zeta^3(1+\zeta\log x)+3\zeta^2(1+\zeta\log x)^2$$
$$+\zeta(1+\zeta\log x)^3+\tfrac{1}{4}(1+\zeta\log x)^4\}-(6\zeta^4+6\zeta^3+3\zeta^2). \qquad (33')$$

The tables for ξ and ζ above show that for $\mu=\frac{4}{3}$, $\xi+\zeta=1$ when $\zeta=0{\cdot}0675$ about, and with this value of ζ (33′) gives $x_s\doteqdot 102$.

The ratio of the mass of the outer region to that of the inner region may be obtained by means of (31′) and is

$$\{(1+\zeta\log x_s)^4-1\}/4\xi=0{\cdot}52.$$

Models with $\xi+\zeta<1$ again represent cases with redundant mass in the outer region.

4. *The Hertzsprung-Russell diagram.*—In the discussion of the present section we use the following results for the photoelectric model with the ratio of the mass in the outer region to that of the inner region $=0{\cdot}088$.

TABLE X

	Increase of radius above	$L_\mu-L_{1/2}$ (mag.)
$\mu=\frac{2}{3}$	2	$-1{\cdot}5$
$\mu=1$	50	$-3{\cdot}5$
$\mu=\frac{4}{3}$	100	$-5{\cdot}0$

In this table the third column gives to the nearest 0·25 magnitude the difference between the bolometric luminosity L_μ when the inner region has molecular weight μ and the bolometric luminosity $L_{1/2}$ when the molecular weight is assumed to be $\frac{1}{2}$ throughout; that is effectively at a time of initial condensation of a star of such a mass. The luminosity L_μ is obtained by treating the inner region as that of a main-sequence star, and can therefore be read off the normal mass-luminosity curves provided

$$2{\cdot}72\ (\text{Mass of inner region}/M_1) \qquad (40)$$

is used instead of the total mass of the star. In the present case, however, (40) differs very little from the total mass, for we have

$$\text{Mass of inner region}=\text{Total Mass}\div 1{\cdot}088,$$

while for both $\mu=\frac{2}{3}$ and $\mu=1$ we have $M_1=2{\cdot}6$, from the solution given by Bondi, and for $\mu=\frac{4}{3}$, $M_1=2{\cdot}7$. So that the mass to be used is 0·972 (total mass) in the first case, and 0·936 (total mass) in the second. These figures show that the correction is unimportant (about 0·2 mag. in the second case) and the results tabulated above accordingly correspond closely to the difference between the luminosities of two main-sequence stars of the same mass but of molecular weights μ and $\frac{1}{2}$. This result applies to values of the total mass in the range $2M_\odot$ to $5M_\odot$, which covers the various examples considered later.

For the electron-scattering model, when the ratio of the mass of the outer region to that of the inner region is taken as 0·52, the corresponding results are as follows:—

TABLE XI

	Increase of radius above main sequence (factor)	$L_\mu-L_{1/2}$ (mag.)
$\mu=1$	4	$-2{\cdot}7$
$\mu=\frac{4}{3}$	50	$-3{\cdot}8$

As before, the value given by (40) has to be used instead of the total mass in reading off L_μ from the normal mass-luminosity curves. In the present case

$$\text{Mass of inner region} = \text{Total mass} \div 1{\cdot}52$$

and for $\mu = 1$, $M_1 = 2{\cdot}35$, while for $\mu = \frac{4}{3}$, $M_1 = 2{\cdot}6$. Thus only 0·77 of the total mass is to be used when $\mu = 1$, and a proportion 0·70 when $\mu = \frac{4}{3}$. These factors appear cubed in calculating L_μ and lead to reductions between 0·8 mag. and 1·2 mag.

In Fig. 1 the points $X_{1/2}$, $X'_{1/2}$, $X''_{1/2}$, $X'''_{1/2}$ lie on the theoretical curve obtained by plotting the positions in the Hertzsprung-Russell diagram of the luminosity of stars with $\mu = \frac{1}{2}$ throughout as functions of their masses. The points P_μ, E_μ represent the positions according to the photoelectric and electron-scattering models respectively of a star initially at $X_{1/2}$ that has evolved to a stage at which its inner region has molecular weight μ. The values of μ adopted for the plotted points are $\mu = \frac{2}{3}$, 1 and $\frac{4}{3}$ for the points P_μ; and $\mu = 1$ and $\frac{4}{3}$ for the points E_μ. The positions of these points can be obtained immediately by means of the above results, although the appropriate bolometric corrections have to be introduced to obtain M_{pv}. In this connection, the temperature scale given by Kuiper * was used for determining the changes of spectral type. The points P'_μ, E'_μ refer in a similar way to a star initially at $X'_{1/2}$; the points P''_μ, E''_μ refer to a star initially at $X''_{1/2}$; and the points P'''_μ, E'''_μ refer to a star initially at $X'''_{1/2}$.

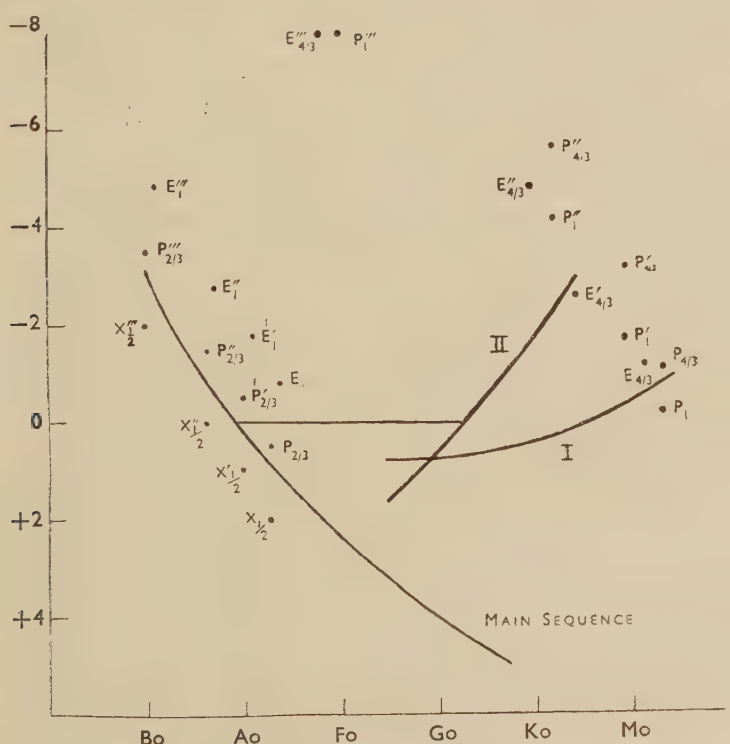

FIG. 1.—*Luminosity-type diagram.*

It will be noticed that in this figure there is, as we may term it, a corridor free from marked points, that lies between the main sequence and the chain of points stretching from P'''_1, $E'''_{4/3}$ to P_1, $E_{4/3}$. This is, however, a selection effect arising from the particular models represented by the diagram. The region could be filled by choosing other values for the ratio of the mass in the outer region to that of

* G. P. Kuiper, *Ap. J.*, **88**, 429, 1938.

the inner region. If this ratio is reduced, the points in the chain move towards the main sequence, whereas if the ratio is increased the points near the main sequence move into the vacant region. Models corresponding to points in this corridor can also be obtained by taking other values of μ. But such cases must be exceptional, as they are found to require very special values of μ. For instance, in the photoelectric case as μ increases above $\frac{2}{3}$, the points do not move across the diagram at anything approaching uniform rate. They stay near the main sequence until μ rises nearly to 1 and then move rapidly away from the main sequence to join the chain of points far to the right. Thus if the representative point of a star falls within this corridor, it will in general do so, not because the value of μ is different from those for the cases plotted in Fig. 1 but because the mass-ratio differs from those adopted for these cases.

It is a satisfactory result of the theory that the main sequence is well-contained between the points corresponding to $\mu=\frac{1}{2}$ and those for $\mu=\frac{2}{3}$. As the time required for μ to change, owing to the conversion of hydrogen to helium by nuclear reactions, from $\frac{1}{2}$ to $\frac{2}{3}$ is considerably greater than the time for an increase from $\frac{2}{3}$ to $\frac{4}{3}$, it follows that the position of a star in the diagram must be in the main sequence throughout most of its lifetime. On the other hand, because the luminosity is far higher in the later stages of evolution, there will be a powerful selection effect leading to stars with μ between $\frac{2}{3}$ and $\frac{4}{3}$ receiving undue prominence and frequency in formal statistical studies of faint stellar populations.

The curves corresponding to the red-giant sequences for the so-called * populations of Types I and II are also shown in Fig. 1. The positions of these curves suggest, in correspondence with the theoretical results of this paper, that the differences giving rise to these populations must represent selection effects. Apart from direct observational selection effects there are likely to be important physical selection effects, such as a correlation of the distribution of the stars and their masses. There is also the possibility of some correlation between the age of a star and its mass; that is, different masses may have statistically different distributions with respect to age. Such questions as these obviously do not fall wholly within the scope of the theory of stellar structure, as they necessarily involve investigation of the origin and evolution of the populations in question. Nevertheless, a few general remarks on such matters may be permitted in concluding the present section.

In the first place, we notice that the time required for evolution from $X_{1/2}$ to a state with $\mu=\frac{4}{3}$ is nearly 4×10^9 years. The corresponding times for evolution from $X'_{1/2}$, $X''_{1/2}$, $X'''_{1/2}$ to $\mu=\frac{4}{3}$ are 10^9 years, 3×10^8 years, and 10^7 years respectively. This last interval (10^7 years) is so short that we should not expect, in a general statistical survey, to find many stars lying near P_1''' or $P_{4/3}'''$. Special cases may occur, however, and are indeed observed; for example, Canopus lies near P_1''', and the giant component of ζ Aurigae lies near $P_{4/3}'''$.

The head of the giant sequence for a Type I population, that is, the region farthest from the main sequence, arises from stars initially near $X_{1/2}$. Indeed, the whole of the Type I giant sequence can be explained in terms of the evolution of stars that are near $X_{1/2}$ at the time of their condensation. Thus it appears that stars of Type I population must be about 4×10^9 years old. This estimate affords a definite lower limit to the age of the Galaxy, and also of other galaxies possessing Type I populations.

* W. Baade, *Ap. J.*, **100**, 137, 1944.

The lower part of the Type II sequence also requires stars evolving from near $X_{1/2}$. But the head of this sequence, towards which the stars must evolve, correspond to stars commencing from between $X'_{1/2}$ and $X''_{1/2}$. This suggests that the members of a Type II population are not of uniform age. The stars near the base of the sequence must be about 4×10^9 years old, whereas the stars near the head must have ages rather less than 10^9 years.

5. *Uniform and non-uniform stars.*—We now enquire into the physical factors determining the difference between these two classes of star, namely those of uniform and non-uniform compositions. Since hydrogen is converted into helium within the core, it is essential in the case of uniform stars that a general mixing process shall be available to preserve the uniformity. Since the region outside the core is stable against ordinary convection, there must be other causes at work to produce the mixing. To consider this, we distinguish two possibilities.

(*a*) To begin with, let it be supposed that there is no general mixing process. Then the helium produced by the nuclear reactions is distributed only through the core and cannot enter the radiative zone, nor can hydrogen enter the core from outside. Since the core contains only about a tenth of the whole mass, this means that generation of energy would cease after the hydrogen in only a small fraction of the whole star had been used up. At this stage the star must abandon the equilibrium state.

For the course of development beyond this stage, it was suggested by Gamow that evolution through a series of non-equilibrium states takes the star to a new equilibrium configuration possessing an isothermal core surrounded by a thin shell in which all the energy is then generated, while outside this shell is a radiative region with an approximately constant energy flux. In this way it was thought possible for a star to continue using its hydrogen without any actual mixing being necessary. The present authors, in agreement with Chandrasekhar's conclusions, and with those of recent calculations by C. M. and H. Bondi, consider that no such equilibrium configurations can exist if the isothermal core contains more than a few per cent of the total mass. Instead it seems likely that the non-equilibrium state must be of such a nature as to bring the star back to the usual configurations with a convective core. That is, some form of circulatory currents must arise that take the helium out of the core and bring in hydrogen. The fact that an entirely convective star is a possible configuration lends some measure of plausibility to this suggestion. The representative points of stars in which such a process occurs may be expected to depart from the main sequence in a fairly smooth way, as is the case for a Type II population.

A further point worthy of note is that at an early stage, before serious departure from the equilibrium state becomes necessary, a certain amount of hydrogen could for a time be brought within the core if the star adopts a pulsating state, so long as in some phase-range the convective core is extended to include new material. This possibility may have some bearing on the cluster Cepheids.

(*b*) We next consider the possibility of there being a process producing general mixing. In rotating stars it has long been recognized that such a process must occur in some degree. Quite apart from the usual causes suggested by Jeans and Eddington, the conversion of hydrogen to helium must be accompanied by some adjustment in the dimensions of the star, and this must give rise to circulatory currents. The effectiveness of such currents will depend on the rotation rate and, if this is great enough, roughly uniform composition will be

maintained. It seems likely that stars of approximately uniform composition with $\mu > \frac{1}{2}$ may arise in this way.

On the other hand, rotatory effects must be weak in a highly distended atmosphere, and consequently once a giant is formed there is unlikely to be any strong mixing tendency, so that the giant character may be expected to persist. Moreover, a giant could form, even if a strong mixing process were originally present, by the addition at the surface of an appreciable quantity of hydrogen, as a result of accretion, provided this took place in an interval less than the time required for effective mixing. This would produce a transition from a normal uniform star to a giant if the molecular weight in the interior were high enough. The departure of such a star from the main sequence, because of the short time necessarily involved in bringing about the giant state, would have the appearance of being almost discontinuous; for the addition of, say, only half the requisite amount of hydrogen would produce little effect on the radius. It is possible that the Hertzsprung gap in a Type I population may be due to this effect. Also, cases of this kind are more likely to arise for stars representable by the photoelectric model, since these require addition of much less hydrogen (than the electron-scattering model) to bring about a given increase of radius.

It seems possible that stellar populations of Types I and II may differ, in that Type I follows case (*b*) while Type II follows case (*a*). If this is correct, it would mean that the angular momentum of stars of Type I must be systematically greater than of stars of Type II. This, however, would be difficult to test observationally because even if the angular momentum is large the rotational velocities at the surface of a highly distended star must in any case be small.

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RED-GIANT MODELS WITH CHEMICAL INHOMOGENEITIES

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Summary

A stellar model is investigated which contains a difference in chemical composition between the envelope and the interior. The necessary numerical integrations are performed and 15 sample cases for this model are computed through in detail. In some cases the model is found to provide stellar radii sufficiently large for moderate red giants.

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1. *Introduction.*—In recent years a number of stellar models have been computed under the assumption that the chemical composition of the deep interior of the star differs from that in the envelope. Most of these investigations (**1**) have been concerned with the depletion of hydrogen by nuclear processes, and consequently they have taken the transition in chemical composition to be located in the deep interior, on the edge of a convective or isothermal core. On the other hand, it has also been considered (**2**) that hydrogen might be enriched in the envelope of a star by accretion. In this case the zone of transition is expected to lie in the outer part of the star.

When these models were applied to stars with a given mass and central temperature, the resulting radii were generally found to be larger than those obtained for chemically homogeneous stars. Thus it appeared possible that these models should represent giants of intermediate spectral type. In all the cases which had been computed through in detail, however, the increase in radius was less than a factor four and hence insufficient to represent late-type giants (**3**). Nevertheless it had been suggested (**2**) that a more detailed consideration of the consequences of chemical inhomogeneities might lead to models with still larger radii.

The purpose of this paper is to investigate further the possibility of large stellar radii arising from chemical inhomogeneities. For this purpose there are derived in detail a series of examples for a stellar model containing a change in chemical composition in its radiative part (**4**).

2. *Assumptions and Basic Equations.*—In accordance with the aim described above, the model to be computed consists of three parts: (i) a convective core; (ii) an intermediate radiative zone with a composition equal to that of the core; and (iii) a radiative envelope differing in composition from the core.

To simplify the computations, the following approximating assumptions are used:

(*a*) The transition in chemical composition occurs discontinuously at a spherical interface; from the centre to the interface the composition (X_i, Y_i, Z_i) is constant; similarly, from the interface to the surface the composition (X_e, Y_e, Z_e) is constant.

(*b*) Radiation pressure and degeneracy are negligible.

(*c*) In the convective core $\gamma = \frac{5}{3}$.

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(*d*) The absorption coefficient is given by (**5**),

$$\kappa=\kappa_0\rho^{0.75}T^{-3.5} \qquad \text{with} \qquad \kappa_0=10^{25}Z(1+X)^{0.75}.$$

(*e*) The entire energy generation from nuclear sources occurs within the convective core.

To facilitate the integrations, the dimensionless variables p, t, q, x, l and j are used. They are defined by

$$P=p\frac{GM^2}{4\pi R^4}, \qquad T=t\mu_s\frac{H}{k}\frac{GM}{R}, \qquad M_r=qM, \qquad r=xR, \tag{1}$$

$$l=\frac{\mu}{\mu_s}, \qquad j=\frac{Z}{Z_s}\left(\frac{(1+X)\mu}{(1+X_s)\mu_s}\right)^{0.75}. \tag{2}$$

Here the subscript s refers to a standard composition. In the present paper the standard chosen is the composition in the envelope of the star ($s=e$). Hence according to assumption (*a*), in the envelope:

$$l_e=1, \qquad j_e=1; \tag{3}$$

but in the core and intermediate zone:

$$l_i=\frac{\mu_i}{\mu_e}=\frac{2X_e+\frac{3}{4}Y_e+\frac{1}{2}Z_e}{2X_i+\frac{3}{4}Y_i+\frac{1}{2}Z_i}, \qquad j_i=\frac{Z_i}{Z_e}\left(\frac{(1+X_i)\mu_i}{(1+X_e)\mu_e}\right)^{0.75}. \tag{4}$$

With these dimensionless variables, the basic equations for the radiative parts are

$$\frac{dp}{dx}=-l\frac{pq}{tx^2}, \qquad \frac{dq}{dx}=+l\frac{px^2}{t}, \qquad \frac{dt}{dx}=-ljC\frac{p^{1.75}}{t^{8.25}x^2}, \tag{5}$$

where the parameter C is a constant throughout the star and is defined by

$$C=\frac{3\kappa_{0e}}{4ac}\left(\frac{k}{\mu_eHG}\right)^{7.5}\left(\frac{1}{4\pi}\right)^{2.75}\frac{LR^{1.25}}{M^{5.75}}. \tag{6}$$

Finally, the homology invariant quantities U, V and n are needed for the fitting procedure. They can be expressed in terms of the dimensionless variables by

$$U=+\frac{r}{M_r}\frac{dM_r}{dr}=l\frac{px^3}{tq}, \qquad V=-\frac{r}{P}\frac{dP}{dr}=l\frac{q}{tx}, \tag{7}$$

$$n+1=\frac{T}{P}\frac{dP}{dT}=\frac{1}{j}\frac{qt^{8.25}}{Cp^{1.75}}. \tag{8}$$

Equations (7) hold throughout the star, whereas equation (8) holds only in the radiative parts; in the convective core, $n+1=2.5$.

In the convective core the basic equations can be reduced to the polytropic equation of index 1.5. For this equation the particular solution that has no singularity at the centre is available in tabular form (**6**). For the radiative envelope, solutions are needed which satisfy equations (5) with conditions (3) and which are regular at the surface. Four such solutions are available in tabular form (**4**), each corresponding to a specific value of C. Hence, new integrations are needed here only for the intermediate radiative zone.

3. *Integrations for Intermediate Zone.*—In the intermediate radiative zone, equations (5) hold with conditions (4). Solutions of these equations were obtained by numerical integration. For these integrations the following logarithmic variables were used.

$$\lambda=\log\frac{p}{p_0}, \qquad \psi=\log\frac{q}{q_0}, \qquad \tau=\log\frac{t}{t_0}, \qquad y=\log\frac{x}{x_0}. \tag{9}$$

The four constants denoted by a subscript zero were chosen so as to fulfil the three conditions

$$l_i \frac{q_0}{t_0 x_0} = 1, \qquad l_i \frac{p_0 x_0^3}{t_0 q_0} = 1, \qquad l_i j_i C \frac{p_0^{1.75}}{t_0^{9.25} x_0} = 1. \tag{10}$$

With this choice for the constants, the basic differential equations (5) become

$$\left.\begin{aligned} \log\left(-\frac{d\lambda}{dy}\right) &= \psi - \tau - y, \\ \log\left(+\frac{d\psi}{dy}\right) &= \lambda - \tau - \psi + 3y, \\ \log\left(-\frac{d\tau}{dy}\right) &= 1{\cdot}75\lambda - 9{\cdot}25\tau - y. \end{aligned}\right\} \tag{11}$$

Since the integrations were to be performed in an outward direction, the starting values had to be found with the help of the conditions for fitting a radiative zone to a convective core. These fitting conditions may conveniently be expressed in terms of the three homology invariants which now take the form

$$\left.\begin{aligned} \log V &= \psi - \tau - y, \\ \log U &= \lambda - \tau - \psi + 3y, \\ \log (n+1) &= \psi + 8{\cdot}25\tau - 1{\cdot}75\lambda. \end{aligned}\right\} \tag{12}$$

Since there is no discontinuity in the composition between the convective core and the intermediate radiative zone, the three homology invariants must be continuous at this interface.

Correspondingly, the starting values were computed in the following way. The convective core was taken to terminate at an arbitrary point ξ_1, where ξ is the independent variable for the polytrope $n = 1{\cdot}5$. The corresponding values V_1 and U_1 were read from the polytropic tables. Furthermore, $(n+1)_1 = 2{\cdot}5$. Introducing these three values into equations (12) one obtains three conditions for the four starting values λ_1, ψ_1, τ_1 and y_1. One of these may be chosen at will (without restriction on the physical solution), since the three conditions (10) leave open one freedom for the four constants with zero subscript. Here $y_1 = 0$ was chosen. With this choice λ_1, ψ_1 and τ_1 were computed from equations (12). The starting values thus found define a unique solution of the differential equations (11).

TABLE I

Numerical solutions for radiative intermediate zone

y	λ	ψ	τ	U	V	$n+1$
			$\xi_1 = 1{\cdot}1189$			
−0·01	+0·86202	0·20393	+0·20518	2·648	1·020	2·444
·00	·85158	·23033	·20095	2·632	1·070	2·500
+0·01	·84062	·25657	·19662	2·615	1·122	2·556
·02	·82913	·28263	·19217	2·596	1·176	2·613
·03	·81708	·30849	·18761	2·576	1·233	2·669
+0·04	+0·80446	0·33414	0·18294	2·555	1·292	2·726
·05	·79123	·35958	·17813	2·532	1·353	2·764
·06	·77738	·38478	·17321	2·508	1·418	2·840
·07	·76286	·40974	·16815	2·483	1·485	2·897
+0·08	+0·74767	+0·43443	+0·16296	2·456	1·554	2·954

TABLE I—*Continued*

y	λ	ψ	τ	U	V	$n+1$
			$\xi_1=1\cdot1189$			
+0·09	+0·73176	0·45885	0·15762	2·428	1·626	3·011
·10	·71512	·48299	·15215	2·399	1·702	3·067
·11	·69772	·50682	·14653	2·368	1·780	3·124
·12	·67951	·53033	·14075	2·336	1·860	3·180
·13	·66049	·55352	·13482	2·302	1·944	3·236
+0·14	+0·64062	0·57637	0·12873	2·267	2·031	3·291
·15	·61986	·59886	·12248	2·231	2·120	3·346
·16	·59819	·62095	·11606	2·194	2·213	3·401
·17	·57559	·64270	·10947	2·155	2·308	3·455
·18	·55201	·66404	·10270	2·115	2·406	3·510
+0·19	+0·52744	0·68498	0·09575	2·074	2·508	3·564
·20	·50184	·70550	·08862	2·031	2·612	3·618
·21	·47519	·72559	·08131	1·987	2·718	3·671
·22	·44746	·74524	·07381	1·943	2·828	3·725
·23	·41862	·76444	·06613	1·897	2·940	3·779
+0·24	+0·38865	0·78318	0·05825	1·850	3·054	3·834
·25	·35752	·80144	·05019	1·803	3·171	3·889
·26	·32520	·81922	·04194	1·754	3·291	3·946
·27	·29169	·83651	·03351	1·705	3·412	4·005
·28	·25695	·85331	·02491	1·655	3·535	4·066
+0·29	+0·22097	0·86961	0·01613	1·604	3·660	4·130
·30	·18374	·88539	+0·00719	1·553	3·786	4·199
·31	·14524	·90066	−0·00190	1·501	3·914	4·274
·32	·10546	·91541	·01112	1·449	4·041	4·356
·33	·06441	·92964	·02044	1·397	4·170	4·449
+0·34	+0·02207	0·94335	−0·02984	1·345	4·297	4·555
·35	−0·02154	·95653	·03929	1·292	4·424	4·678
·36	·06640	·96919	·04873	1·239	4·549	4·824
·37	·11250	·98132	·05811	1·187	4·671	4·998
·38	·15980	0·99292	·06737	1·134	4·790	5·209
+0·39	−0·20825	1·00400	−0·07645	1·082	4·903	5·467
·40	·25782	1·01456	·08526	1·030	5·010	5·786
·41	·30845	1·02460	·09371	0·979	5·109	6·185
·42	·35997	1·03414	·10173	0·928	5·198	6·680
·43	·41235	1·04317	·10922	0·878	5·277	7·306
+0·44	−0·46544	1·05171	−0·11612	0·830	5·344	8·095
·45	·51913	1·05977	·12238	·783	5·397	9·091
·46	·57328	1·06737	·12797	·737	5·437	10·349
·47	·62777	1·07451	·13288	·693	5·463	11·936
·48	·68244	1·08124	·13713	·651	5·475	13·937
+0·49	−0·73717	1·08755	−0·14076	0·611	5·474	16·456
·50	·79183	1·09347	·14382	·574	5·461	19·617
·51	·84627	1·09903	·14638	·538	5·438	23·572
·52	·90047	1·10426	·14849	·506	5·404	28·513
+0·53	−0·95430	1·10916	−0·15022	0·475	5·363	34·665

TABLE I—*Continued*

y	λ	ψ	τ	U	V	$n+1$
			$\xi_1=1\cdot1204$			
−0·01	+0·86331	0·20525	+0·20530	2·647	1·023	2·450
·00	·85283	·23164	·20106	2·631	1·073	2·500
+0·01	·84185	·25787	·19672	2·614	1·125	2·556
·02	·83032	·28391	·19226	2·595	1·179	2·612
·03	·81824	·30976	·18768	2·575	1·236	2·668
+0·04	+0·80558	0·33540	+0·18299	2·559	1·295	2·725
·05	·79232	·36083	·17817	2·531	1·357	2·781
·06	·77842	·38062	·17322	2·507	1·422	2·837
·07	·76387	·41096	·16815	2·482	1·489	2·893
·08	·74863	·43565	·16293	2·455	1·559	2·950
+0·09	+0·73268	0·46006	+0·15758	2·427	1·631	3·005
·10	·71599	·48418	·15207	2·397	1·707	3·060
·11	·69853	·50800	·14642	2·367	1·785	3·116
·12	·68028	·53150	·14062	2·334	1·866	3·170
·13	·66120	·55467	·13465	2·301	1·950	3·224
+0·14	+0·64126	0·57751	+0·12852	2·266	2·037	3·278
·15	·62044	·59998	·12221	2·230	2·127	3·330
·16	·59870	·62209	·11574	2·192	2·220	3·382
·17	·57602	·64382	·10908	2·153	2·316	3·433
·18	·55236	·66515	·10224	2·113	2·415	3·485
+0·19	+0·52770	0·68608	+0·09522	2·072	2·517	3·533
·20	·50200	·70659	·08799	2·030	2·622	3·581
·21	·47524	·72666	·08057	1·986	2·730	3·628
·22	·44739	·74630	·07294	1·942	2·840	3·674
·23	·41842	·76549	·06510	1·896	2·954	3·718
+0·24	+0·38829	0·78421	+0·05705	1·849	3·070	3·762
·25	·35699	·80247	·04878	1·802	3·189	3·803
·26	·32449	·82024	·04028	1·754	3·311	3·843
·27	·29075	·83754	·03154	1·705	3·436	3·881
·28	·25576	·85433	·02256	1·655	3·563	3·917
+0·29	+0·21948	0·87063	+0·01334	1·605	3·692	3·950
·30	·18190	·88643	+0·00387	1·555	3·824	3·981
·31	·14298	·90172	−0·00587	1·504	3·959	4·009
·32	·10270	·91650	·01589	1·452	4·096	4·033
·33	·06104	·93077	·02619	1·401	4·236	4·054
+0·34	+0·01796	0·94452	−0·03680	1·350	4·378	4·069
·35	−0·02655	·95776	·04772	1·298	4·524	4·079
·36	·07253	·97048	·05899	1·247	4·672	4·081
·37	·12000	·98271	·07063	1·197	4·823	4·074
·38	·16901	0·99442	·08269	1·146	4·979	4·055
+0·39	−0·21960	1·00563	−0·09522	1·097	5·139	4·021
·40	·27181	1·01635	·10829	1·047	5·304	3·969
·41	·32572	1·02658	·12199	0·999	5·477	3·892
·42	·38141	1·03634	·13650	·952	5·660	3·782
·43	·43901	1·04564	·15205	·907	5·857	3·626
+0·44	−0·4987	1·0545	−0·1690	0·86	6·07	3·41
·45	·5606	1·0629	·1880	·82	6·32	3·11
·46	·6253	1·0709	·2103	·78	6·63	2·69
+0·47	−0·6935	1·0786	−0·2386	0·75	7·03	2·11

TABLE I—*Continued*

y	λ	ψ	τ	U	V	$n+1$
			$\xi_1=1\cdot1213$			
−0·01	+0·86407	0·20604	+0·20537	2·647	1·025	2·445
·00	·85358	·23242	·20113	2·631	1·075	2·500
+0·01	·84258	·25865	·19677	2·613	1·127	2·556
·02	·83104	·28468	·19231	2·595	1·181	2·612
·03	·81894	·31053	·18772	2·575	1·238	2·667
+0·04	+0·80626	0·33616	0·18302	2·553	1·298	2·723
·05	·79297	·36158	·17819	2·531	1·360	2·779
·06	·77905	·38677	·17323	2·506	1·424	2·835
·07	·76447	·41170	·16815	2·481	1·491	2·891
·08	·74921	·43638	·16292	2·454	1·561	2·947
+0·09	+0·73323	0·46078	0·15754	2·426	1·634	3·002
·10	·71651	·48489	·15202	2·397	1·710	3·056
·11	·69902	·50870	·14635	2·366	1·788	3·110
·12	·68073	·53220	·14052	2·334	1·869	3·164
·13	·66162	·55536	·13453	2·300	1·954	3·217
+0·14	+0·64164	0·57819	0·12838	2·265	2·041	3·269
·15	·62078	·60066	·12205	2·229	2·131	3·320
·16	·59900	·62276	·11554	2·191	2·224	3·370
·17	·57627	·64447	·10884	2·153	2·321	3·419
·18	·55256	·66580	·10196	2·113	2·420	3·467
+0·19	+0·52785	0·68672	0·09488	2·071	2·523	3·513
·20	·50209	·70721	·08759	2·029	2·628	3·558
·21	·47527	·72728	·08010	1·985	2·736	3·601
·22	·44735	·74691	·07239	1·941	2·848	3·641
·23	·41829	·76609	·06445	1·896	2·962	3·680
+0·24	+0·38811	0·78482	0·05628	1·849	3·080	3·715
·25	·35670	·80307	·04786	1·802	3·200	3·747
·26	·32408	·82085	·03919	1·754	3·324	3·776
·27	·29020	·83815	·03025	1·705	3·451	3·801
·28	·25504	·85495	·02103	1·656	3·580	3·820
+0·29	+0·21857	0·87126	+0·01150	1·606	3·713	3·834
·30	·18075	·88707	+0·00164	1·556	3·850	3·840
·31	·14155	·90238	−0·00857	1·506	3·990	3·837
·32	·10093	·91719	·01917	1·455	4·134	3·823
·33	·05885	·93148	·03022	1·405	4·283	3·795
+0·34	+0·01525	0·94528	−0·04178	1·354	4·437	3·749
·35	−0·02992	·95857	·05395	1·305	4·598	3·680
·36	·07675	·97137	·06685	1·255	4·767	3·583
·37	·12532	·98368	·08067	1·207	4·947	3·447
·38	·17576	0·99551	·09574	1·160	5·143	3·260
+0·39	−0·2283	1·0069	−0·1125	1·12	5·36	3·01
·40	·2832	1·0178	·1319	1·07	5·62	2·66
+0·41	−0·3410	1·0283	−0·1561	1·04	5·95	2·18

TABLE I—*Continued*

y	λ	ψ	τ	U	V	$n+1$
			$\xi_1=1\cdot1250$			
−0·01	+0·86723	0·20944	+0·20566	2·643	1·032	2·446
·00	·85666	·23564	·20139	2·628	1·082	2·500
+0·01	·84559	·26200	·19701	2·610	1·135	2·556
·02	·83396	·28800	·19251	2·591	1·190	2·611
·03	·82178	·31381	·18789	2·571	1·247	2·665
+0·04	+0·80900	0·33941	+0·18315	2·550	1·307	2·720
·05	·79562	·36479	·17827	2·527	1·369	2·775
·06	·78160	·38994	·17327	2·503	1·434	2·829
·07	·76691	·41483	·16813	2·477	1·502	2·881
·08	·75154	·43947	·16280	2·451	1·573	2·933
+0·09	+0·73544	0·46384	+0·15736	2·422	1·646	2·986
·10	·71859	·48786	·15177	2·393	1·722	3·037
·11	·70098	·51164	·14602	2·362	1·801	3·087
·12	·68255	·53510	·14010	2·330	1·884	3·136
·13	·66328	·55823	·13400	2·296	1·969	3·184
+0·14	+0·64315	0·58102	+0·12772	2·262	2·057	3·230
·15	·62212	·60346	·12126	2·225	2·149	3·274
·16	·60015	·62552	·11459	2·188	2·244	3·316
·17	·57722	·64721	·10772	2·149	2·342	3·356
·18	·55330	·66841	·10063	2·110	2·442	3·391
+0·19	+0·52835	0·68930	+0·09330	2·069	2·547	3·423
·20	·50234	·70978	·08573	2·027	2·655	3·451
·21	·47523	·72983	·07791	1·984	2·767	3·475
·22	·44699	·74945	·06980	1·940	2·882	3·492
·23	·41757	·76816	·06139	1·895	3·001	3·502
+0·24	+0·38695	0·78734	+0·05265	1·849	3·124	3·504
·25	·35507	·80560	·04354	1·803	3·251	3·495
·26	·32189	·82339	·03402	1·756	3·384	3·473
·27	·28737	·84071	·02403	1·709	3·521	3·436
·28	·25144	·85755	·01349	1·661	3·665	3·379
+0·29	+0·21403	0·87393	+0·00229	1·614	3·816	3·298
·30	·17507	·88983	−0·00971	1·567	3·977	3·187
·31	·13445	·90526	·02278	1·520	4·150	3·034
·32	·09200	·92024	·03726	1·476	4·340	2·830
·33	·04753	·93479	·05374	1·434	4·555	2·560
+0·34	+0·0007	0·9489	−0·0729	1·40	4·80	2·22
			$\xi_1=1\cdot1400$			
−0·01	+0·87995	0·22228	+0·20683	2·635	1·060	2·448
·00	·86901	·24855	·20244	2·618	1·112	2·500
+0·01	·85755	·27465	·19794	2·600	1·166	2·552
·02	·84561	·30055	·19330	2·581	1·223	2·603
·03	·83308	·32626	·18854	2·560	1·282	2·653
+0·04	+0·81996	0·35175	+0·18364	2·539	1·343	2·703
·05	·80621	·37698	·17860	2·516	1·407	2·752
·06	·79180	·40201	·17344	2·491	1·474	2·800
·07	·77670	·42679	·16810	2·465	1·544	2·847
+0·08	+0·76089	+0·45130	+0·16259	2·438	1·617	2·891

TABLE I—*Continued*

y	λ	ψ	τ	U	V	$n+1$
			$\xi_1=1\cdot1400$			
+0·09	+0·74434	0·47554	+0·15690	2·409	1·693	2·933
·10	·72702	·49948	·15104	2·380	1·772	2·973
·11	·70888	·52312	·14498	2·349	1·854	3·010
·12	·68991	·54644	·13871	2·316	1·940	3·044
·13	·67007	·56943	·13223	2·283	2·029	3·074
+0·14	+0·64932	0·59208	+0·12551	2·248	2·121	3·099
·15	·62762	·61438	·11853	2·212	2·218	3·118
·16	·60494	·63631	·11127	2·175	2·318	3·130
·17	·58124	·65786	·10370	2·136	2·422	3·135
·18	·55648	·67903	·09579	2·097	2·531	3·129
+0·19	+0·53060	0·69980	+0·08750	2·057	2·644	3·113
·20	·50356	·72107	·07877	2·017	2·763	3·082
·21	·47531	·74013	·06954	1·975	2·888	3·034
·22	·44577	·75966	·05968	1·934	3·020	2·964
·23	·41487	·77879	·04907	1·892	3·160	2·868
+0·24	+0·38201	0·79751	+0·03751	1·849	3·311	2·745
·25	·34807	·81577	·02477	1·809	3·475	2·572
+0·26	+0·3124	0·8337	+0·0103	1·77	3·66	2·35
			$\xi_1=1\cdot2000$			
−0·01	+0·92833	0·27270	+0·21127	2·596	1·179	2·461
·00	·91626	·29857	·20640	2·578	1·236	2·500
+0·01	·90360	·32416	·20138	2·559	1·297	2·537
·02	·89031	·34975	·19619	2·537	1·360	2·572
·03	·87640	·37502	·19081	2·515	1·426	2·603
+0·04	+0·86181	0·40006	+0·18524	2·492	1·496	2·631
·05	·84624	·42486	·17945	2·466	1·568	2·657
·06	·83020	·44933	·17345	2·440	1·644	2·676
·07	·81338	·47360	·16719	2·413	1·723	2·683
·08	·79575	·49760	·16053	2·386	1·807	2·686
+0·09	+0·77726	0·52133	+0·15361	2·357	1·895	2·682
·10	·75787	·54475	·14636	2·327	1·988	2·667
·11	·73753	·56787	·13872	2·295	2·085	2·640
·12	·71619	·59066	·13060	2·264	2·188	2·599
·13	·69376	·61315	·12186	2·232	2·297	2·539
+0·14	+0·6702	0·6353	+0·1124	2·20	2·41	2·45

In the above procedure only one arbitrary choice (ξ_1) occurred. Hence, the solutions for the intermediate zone, expressed in terms of the logarithmic variables (9), form a single parameter family. Six particular solutions of this family (for $\xi_1=1\cdot1189$, $1\cdot1204$, $1\cdot1213$, $1\cdot1250$, $1\cdot1400$ and $1\cdot2000$) were derived by numerical integration. They are given in Table I and are represented in the U–V plane in Fig. 1.

An additional solution of this family (for $\xi_1=1\cdot12009$) is available from an earlier integration (5). This particular solution is the only one of this family that is regular at the surface. Solutions with $\xi_1<1\cdot12009$ show a steadily

increasing n and asymptotically behave like isothermals. Solutions with $\xi_1 > 1{\cdot}12009$ terminate their applicable range where n drops below 1·5. Such a solution would have to be continued with the adiabatic instead of the radiative equations. This would introduce an additional convection zone into the model. Such models have not been considered here.

4. *Fitting at Discontinuity of Composition.*—For the construction of the model here considered, it is necessary to fit an integration for the intermediate zone to

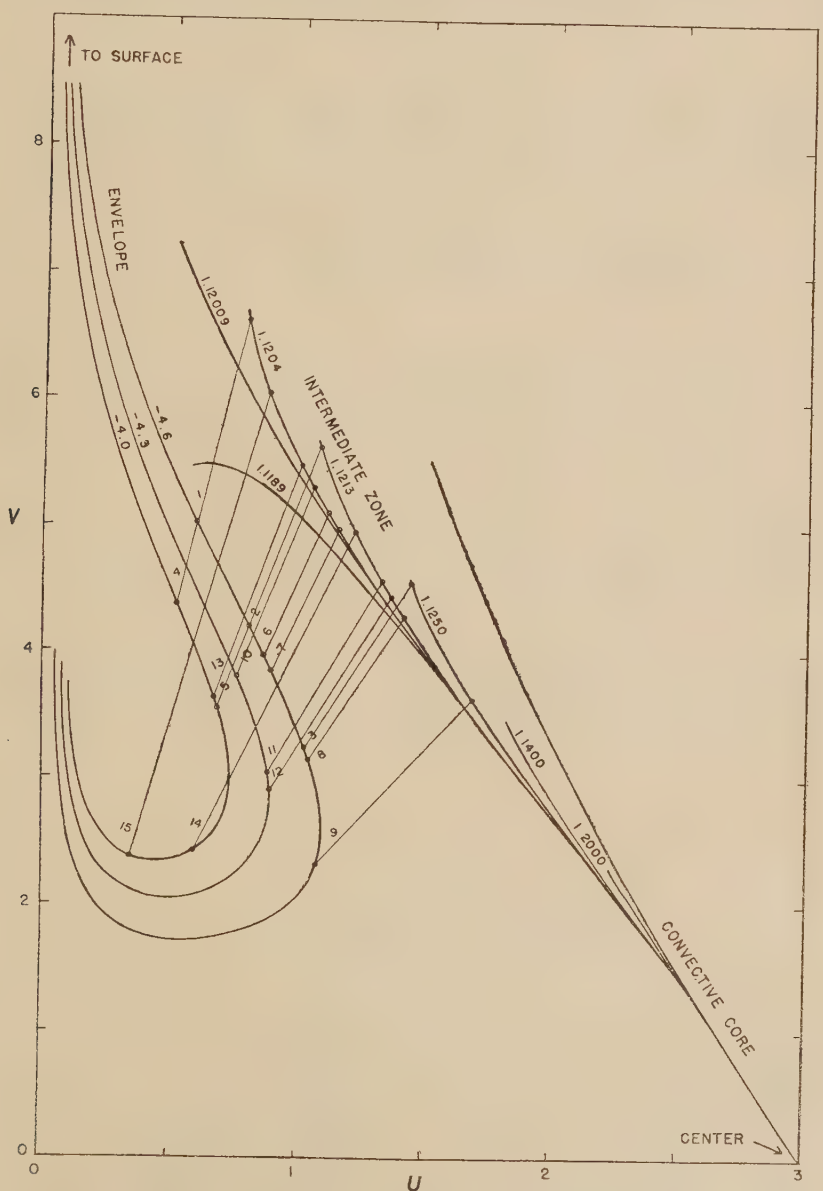

FIG. 1.—*Representation of the models in terms of the homology invariants U and V.*

The numbers on the envelope solutions indicate the value of log C, the numbers on the curves for the intermediate zone give the values of ξ_1, and the numbers on the straight lines representing the discontinuity correspond to the enumeration of sample cases in Tables II and V.

TABLE

Mathematical characteristics of 15 *sample*

Case	1	2	3	4	5	6	7
ξ_1	1·1204	1·1213	1·1250	1·1204	1·1213	1·1204	1·1204
θ_1	0·8092	0·8089	0·8078	0·8092	0·8089	0·8092	0·8092
U_1	2·631	2·631	2·628	2·631	2·631	2·631	2·631
V_1	1·073	1·075	1·082	1·073	1·075	1·073	1·073
$y_2=\log(x_2/x_1)$	0·4600	0·4000	0·3300	0·4600	0·4000	0·3880	0·3800
$\log(p_1/p_2)$	1·4781	1·1368	0·8092	1·4781	1·1368	1·0623	1·0218
$\log(q_2/q_1)$	0·8393	0·7854	0·6992	0·8393	0·7854	0·7718	0·7628
$\log(t_1/t_2)$	0·4114	0·3330	0·2551	0·4114	0·3330	0·2938	0·2838
U_{2i}	0·784	1·071	1·434	0·784	1·071	1·106	1·146
V_{2i}	6·626	5·620	4·555	6·626	5·620	5·106	4·979
$(n+1)_{2i}$	2·691	2·662	2·560	2·691	2·662	4·029	4·055
$\log C$	−4·6	−4·6	−4·6	−4·0	−4·0	−4·6	−4·6
U_{2e}	0·592	0·799	1·018	0·517	0·677	0·858	0·886
V_{2e}	5·010	4·194	3·235	4·373	3·551	3·960	3·847
$(n+1)_{2e}$	4·471	4·331	4·101	4·459	4·305	4·282	4·257
l_i	1·323	1·340	1·408	1·515	1·582	1·290	1·294
j_i	1·661	1·627	1·602	1·657	1·617	1·063	1·050
$\log(1/x_2-1)$	0·5482	0·6780	0·8448	0·8054	1·0071	0·7168	0·7358
$\log p_2$	1·4830	1·9613	2·4873	2·2782	2·9860	2·0931	2·1556
$\log q_2$	−0·1078	−0·1809	−0·3111	−0·1344	−0·2427	−0·2077	−0·2216
$\log t_2$	−0·1511	−0·0425	+0·0820	+0·0934	+0·2548	−0·1019	+0·0028
$x_1=r_1/R$	0·077	0·069	0·058	0·047	0·036	0·066	0·065
$x_2=r_2/R$	0·221	0·173	0·125	0·135	0·090	0·161	0·155
$\log p_c$	3·1909	3·3284	3·5282	3·9861	4·3531	3·3852	3·4072
$\log t_c$	0·3522	0·3826	0·4298	0·5967	0·6699	0·3738	0·3785

TABLE III

Values of l_i and j_i *for* $Y_i=Y_e=0$

X_e \ X_i	0·0	0·2	0·4	0·6	0·8	0·0	0·2	0·4	0·6	0·8
0·0	1·000					1·000				
0·1	1·300					1·259				
0·2	1·600	1·000		l_i		1·551	1·000		j_i	
0·3	1·900	1·188				1·899	1·224			
0·4	2·200	1·375	1·000			2·339	1·508	1·000		
0·5	2·500	1·563	1·136			2·937	1·892	1·254		
0·6	2·800	1·750	1·273	1·000		3·804	2·453	1·626	1·000	
0·7	3·100	1·938	1·409	1·107		5·230	3·373	2·236	1·375	
0·8	3·400	2·125	1·546	1·214	1·000	8·057	5·194	3·444	2·118	1·000
0·9	3·700	2·313	1·682	1·321	1·088	16·484	10·629	7·047	4·334	2·046
1·0	4·000	2·500	1·818	1·429	1·177	∞	∞	∞	∞	∞

II

models with chemical inhomogeneities

8	9	10	11	12	13	14	15
1·1213	1·1250	1·1204	1·1213	1·1213	1·1204	1·1213	1·1204
0·8089	0·8078	0·8092	0·8089	0·8089	0·8092	0·8089	0·8092
2·631	2·628	2·631	2·631	2·631	2·631	2·631	2·631
1·075	1·082	1·073	1·075	1·075	1·073	1·075	1·073
0·3300	0·2770	0·4000	0·3480	0·3400	0·4100	0·3700	0·4388
0·7947	0·5945	1·1246	0·8745	0·8383	1·1785	0·9789	1·3442
0·6991	0·6169	0·7848	0·7235	0·7129	0·7550	0·7513	0·8218
0·2313	0·1847	0·3093	0·2526	0·2429	0·3231	0·2818	0·3680
1·405	1·675	1·047	1·314	1·354	0·999	1·207	0·868
4·283	3·621	5·304	4·565	4·437	5·477	4·947	6·046
3·795	3·399	3·969	3·696	3·749	3·892	3·447	3·441
−4·6	−4·6	−4·3	−4·3	−4·3	−4·0	−4·0	−4·0
1·033	1·071	0·751	0·873	0·882	0·663	0·589	0·341
3·148	2·315	3·805	3·031	2·890	3·635	2·413	2·374
4·079	3·824	4·305	4·123	4·087	4·289	4·119	4·289
1·360	1·564	1·394	1·506	1·535	1·507	2·050	2·547
1·075	1·125	1·085	1·115	1·090	1·102	1·195	1·246
0·8617	1·0573	0·8376	1·0132	1·0507	0·9852	1·4826	1·8073
2·5342	3·0096	2·4529	2·9862	3·0884	2·9143	4·2703	5·0168
−0·3264	−0·5156	−0·2145	−0·3436	−0·3736	−0·2295	−0·5518	−0·6891
+0·0935	+0·2139	+0·1019	+0·2284	+0·2533	+0·2382	+0·5624	+0·7406
0·057	0·043	0·051	0·039	0·037	0·037	0·014	0·005
0·121	0·081	0·127	0·088	0·082	0·094	0·032	0·015
3·5592	3·8358	3·8073	4·0910	4·1570	4·3226	5·4795	6·5908
0·4169	0·4913	0·5031	0·5731	0·5883	0·6532	0·9363	1·2005

TABLE IV

Values of l_i and j_i for $Z_i = Z_e = 0{\cdot}1$

X_e \ X_i	0·0	0·2	0·4	0·6	0·8	0·0	0·2	0·4	0·6	0·8
0·0	1·000					1·000				
0·1	1·172					1·049				
0·2	1·345	1·000		l_i		1·089	1·000		j_i	
0·3	1·517	1·128				1·123	1·031			
0·4	1·690	1·256	1·000			1·151	1·057	1·000		
0·5	1·862	1·385	1·102			1·177	1·080	1·021		
0·6	2·034	1·513	1·204	1·000		1·197	1·100	1·043	1·000	
0·7	2·207	1·641	1·306	1·085		1·216	1·117	1·057	1·016	
0·8	2·379	1·769	1·408	1·169	1·000	1·233	1·132	1·071	1·029	1·000
0·9	2·552	1·897	1·510	1·254	1·072	1·247	1·146	1·084	1·042	1·011

an integration for the envelope. At the interface between the intermediate zone and the envelope the physical quantities P, T, M_r and r must be continuous, whereas the chemical composition is assumed to change discontinuously. Hence, according to equations (1) and (2), the dimensionless quantities p, t, q and x are continuous at the interface, whereas the quantities l and j jump from their interior values, given by (4), to their envelope values, given by (3).

For the fitting, it is again convenient to use the homology invariant quantities U, V and n. According to equations (7) and (8) these quantities undergo discontinuous changes at the interface because of the discontinuities in l and j. If U_{2e}, V_{2e} and n_{2e} are the values just outside the interface and U_{2i}, V_{2i} and n_{2i} the values just inside the interface, one finds from equations (7) and (8):

$$\frac{U_{2i}}{U_{2e}} = \frac{l_i}{l_e} = l_i, \quad \frac{V_{2i}}{V_{2e}} = \frac{l_i}{l_e} = l_i, \quad \frac{n_{2e}+1}{n_{2i}+1} = \frac{j_i}{j_e} = j_i. \tag{13}$$

From the first two of these equations it follows that

$$\frac{U_{2i}}{V_{2i}} = \frac{U_{2e}}{V_{2e}}. \tag{14}$$

This relation shows that the discontinuity may be represented in the U–V plane (Fig. 1) by a jump from a point on a solution for the intermediate zone to a point on an envelope solution, where the jump has to occur along a straight line pointing to the origin of the U–V plane.

5. *Construction of Complete Model.*—A complete model may now be assembled in the following sequence, with the help of the U–V plane. Start at the point representing the centre ($U=3$, $V=0$); follow the convective core solution; terminate the convective core at an arbitrary point and switch to the corresponding radiative solution for the intermediate zone; follow this solution; terminate the intermediate zone at an arbitrary point; jump from this point to a point on an envelope solution; in this jump choose the particular envelope solution (that is, the value of C) arbitrarily, but determine the starting-point on the envelope solution so that equation (14) is fulfilled; follow the envelope solution to the surface ($U=0$, $V=\infty$).

The above procedure involves three arbitrary choices. This arbitrariness is equivalent to the freedom of choosing (*a*) the place in the star at which the composition changes, (*b*) the amount of change in the molecular weight, and (*c*) the amount of change in the absorption coefficient.

After a model has been assembled in the U–V plane, all the dimensionless quantities, at any point in the star, can be computed from the tables of the integrations used in the model (**7**). The essential dimensionless quantities are given in Table II for 15 examples of the type of model here considered. The 15 cases differ from one another in the choice of the three arbitrary quantities (ξ_1 for the termination of the core, y_2 for the termination of the intermediate zone and C for the particular envelope solution).

Before the physical characteristics of a model star can be derived from the dimensionless quantities of Table II, it is necessary to interpret the two parameters l_i and j_i in terms of the composition in the envelope and in the interior. For a given pair of values for l_i and j_i, equations (4) provide only two conditions between the four free composition parameters X_i, Y_i, X_e and Y_e. Hence additional assumptions have to be made regarding the composition. For the first five of the

cases listed in Table II it has been assumed that the helium content is negligible throughout the star ($Y_i = Y_e = 0$). This assumption makes these five cases directly comparable with the results of earlier computations (**2**). For the remaining ten cases it has been assumed that the heavy elements (other than *H* and *He*) provide 10 per cent of the mass throughout the star ($Z_i = Z_e = 0{\cdot}1$). This assumption corresponds to the effects of the carbon cycle, which does not change the abundance of the heavy elements, but depletes hydrogen and enriches helium in the interior.

Under each of the above assumptions the composition in the interior and in the envelope is completely fixed by X_i and X_e respectively. Hence l_i and j_i may be computed as functions of X_i and X_e according to equations (4). The results are given in Tables III and IV. These tables show the limits of the range for l_i resulting from the limited range in the molecular weight μ. These limits imply, according to the first two of equations (13), that in the U–V plane the jump representing the discontinuity cannot be arbitrarily large. Furthermore, Table IV shows that, for the assumption of constant percentage of heavy elements, j_i never differs greatly from one. Therefore, according to the last of equations (13), a corresponding jump in the U–V plane must have a starting-point and an end-point which differ not too much in their n-values.

By an inverse use of Tables III and IV the values of X_i and X_e are determined from the values of l_i and j_i given in Table II for each of the 15 cases. The resulting compositions are listed in the first eight lines of Table V.

6. *Results.*—The physical characteristics of a star, built according to the model here considered, can be computed from the dimensionless quantities in Table II and the chemical compositions in Table V, if the mass of the star is given. Here we shall use, as representative of heavy stars, a mass of ten solar masses. Furthermore, assuming that the carbon cycle provides the energy generation in the core, the necessary central temperature for a giant is estimated to be approximately 30 million degrees (**8**).

With this choice for M and T_c the essential physical quantities have been computed corresponding to the 15 sample cases listed in Table II. The resulting data are given in Table V. The ninth line of this table shows that, for the majority of the 15 cases, the computed radii are smaller than the radii of heavy red giants. Only for cases 5, 14 and 15 do the radii approach the right size. In these cases the density distribution consists of a small, fairly dense core and an extended, tenuous envelope.

The values of the absorption coefficient, listed in Table V, were derived for the interior of these models from the simple power law assumed throughout the present computations. They are smaller than the electron-scattering coefficient, in the worst case by a factor over ten. Nevertheless it appears doubtful whether the introduction of electron scattering will modify the models appreciably, since electron scattering is less important in the envelope than in the interior. Furthermore, for smaller masses the necessary correction decreases; thus if case 15 is applied to a star of three solar masses (instead of ten) electron scattering nowhere exceeds one-half of the assumed opacity.

To facilitate the comparison with observations one may compute the absolute magnitudes and the spectral types of the model stars here derived. This has been done only for the extreme case 15. For the data given in Table VI, case 15 has not only been applied to stars of ten solar masses (as in Table V), but also to stars of five and three solar masses, all with a central temperature of 30 million degrees,

TABLE

Physical characteristics of 15 sample models with chemical inhomogeneities, applied

Meaning of subscripts : 1, interface between convective core and radiative zone; 2, interface core and intermediate zone;

Case	1	2	3	4	5	6	7
X_i	0·33	0·31	0·22	0·15	0·06	0·20	0·52
Y_i	0·00	0·00	0·00	0·00	0·00	0·70	0·38
Z_i	0·67	0·69	0·78	0·85	0·94	0·10	0·10
μ_i	1·00	1·04	1·20	1·39	1·69	1·03	0·73
X_e	0·55	0·53	0·45	0·39	0·29	0·42	0·83
Y_e	0·00	0·00	0·00	0·00	0·00	0·48	0·07
Z_e	0·45	0·47	0·55	0·61	0·71	0·10	0·10
μ_e	0·75	0·77	0·86	0·92	1·07	0·80	0·57
$R/R_\odot$	13·0	14·3	17·8	28·0	39·4	14·6	10·5
$r_1/R_\odot$	1·0	1·0	1·0	1·3	1·4	1·0	0·7
$r_2/R_\odot$	2·9	2·5	2·2	3·8	3·5	2·3	1·6
$M_{r_1}/M_\odot$	1·13	1·08	0·98	1·06	0·94	1·05	1·04
$M_{r_2}/M_\odot$	7·80	6·59	4·89	7·34	5·72	6·20	6·00
$\log \bar{\rho}$	− 2·19	− 2·32	− 2·60	− 3·19	− 3·64	− 2·34	− 1·91
$\log \rho_c$	+ 0·29	+ 0·28	+ 0·17	− 0·10	− 0·24	+ 0·30	+ 0·75
$\log \rho_1$	+ 0·15	+ 0·14	+ 0·03	− 0·24	− 0·38	+ 0·17	+ 0·61
$\log \rho_{2i}$	− 0·91	− 0·66	− 0·53	− 1·30	− 1·19	− 0·60	− 0·13
$\log \rho_{2e}$	− 1·04	− 0·79	− 0·68	− 1·48	− 1·38	− 0·71	− 0·24
$T_1 \cdot 10^{-6}$	24·3	24·3	24·2	24·3	24·3	24·3	24·3
$T_2 \cdot 10^{-6}$	9·4	11·3	13·5	9·4	11·3	12·3	12·6
$\log \kappa_1$	− 0·82	− 0·82	− 0·87	− 1·05	− 1·14	− 1·66	− 1·25
$\log \kappa_{2e}$	− 0·39	− 0·46	− 0·60	− 0·63	− 0·78	− 1·24	− 0·84
$\log L/L_\odot$	4·67	4·69	4·88	5·42	5·69	5·50	4·49

and all with the following composition : 0 per cent hydrogen and 90 per cent helium in the interior and 90 per cent hydrogen and 0 per cent helium in the envelope. Table VI indicates that case 15 may be taken possibly to represent the internal structure of moderate red giants. However, even this case (with its very large variation in composition from interior to envelope) does not give radii as large as those of the more extreme red giants, if a central temperature adequate for the carbon cycle is assumed.

7. *Speculations.*—The above results raise the question whether larger radii could be obtained with the type of model here considered, if the choice of values for the three free parameters were to be extended beyond the range covered by the 15 cases of Table II. Two directions for extending the parameters might be considered.

First, larger values of C could be used for the envelope solutions. Corresponding envelope integrations are not available at present. Extrapolation, however, from the existing integrations makes it appear likely that larger radii could indeed be obtained, if appreciably larger C values were employed—even for cases in which the envelope contained a small part of the total mass as had been suggested earlier (**2**). On the other hand, for several red giants approximate values of mass, luminosity and radius have been derived from observations (**9**). From these data

V

to stars of ten times the solar mass and a central temperature of 30 million degrees

of chemical discontinuity between intermediate zone and envelope; *i*, interior consisting of *e*, envelope; *c*, centre.

8	9	10	11	12	13	14	15
0·21	0·04	0·16	0·05	0·35	0·16	0·02	0·00
0·69	0·86	0·74	0·85	0·55	0·74	0·88	0·90
0·10	0·10	0·10	0·10	0·10	0·10	0·10	0·10
1·02	1·29	1·09	1·28	0·86	1·08	1·34	1·38
0·49	0·39	0·45	0·36	0·84	0·54	0·65	0·90
0·41	0·51	0·45	0·54	0·06	0·36	0·25	0·00
0·10	0·10	0·10	0·10	0·10	0·10	0·10	0·10
0·75	0·82	0·78	0·85	0·56	0·71	0·65	0·54
15·1	19·6	19·1	24·5	16·7	24·6	43·2	66·0
0·9	0·8	1·0	1·0	0·6	0·9	0·6	0·3
1·8	1·6	2·4	2·2	1·4	2·3	1·4	1·0
0·94	0·74	1·00	0·86	0·82	0·95	0·50	0·30
4·72	3·05	6·10	4·53	4·23	5·90	2·81	2·00
− 2·39	− 2·73	− 2·70	− 3·02	− 2·52	− 3·02	− 3·76	− 4·31
+ 0·41	+ 0·34	+ 0·28	+ 0·20	+ 0·76	+ 0·35	+ 0·62	+ 1·01
+ 0·27	+ 0·20	+ 0·14	+ 0·06	+ 0·62	+ 0·21	+ 0·48	+ 0·87
− 0·29	− 0·21	− 0·68	− 0·56	+ 0·02	− 0·65	− 0·22	− 0·10
− 0·42	− 0·41	− 0·82	− 0·74	− 0·16	− 0·83	− 0·53	− 0·51
24·3	24·2	24·3	24·3	24·3	24·3	24·3	24·3
14·2	15·8	11·9	13·5	13·9	11·5	12·7	10·4
− 1·58	− 1·68	− 1·69	− 1·79	− 1·28	− 1·64	− 1·48	− 1·20
− 1·22	− 1·40	− 1·26	− 1·42	− 0·92	− 1·20	− 1·10	− 0·73
5·25	5·43	5·56	5·73	4·48	5·40	4·78	3·90

C may be computed with the help of equation (6), if a composition for the envelope is assumed. The C values thus found (even if the optimum composition for large C is assumed) in no case exceed by more than a factor two the maximum value here used. Hence, the observational data do not seem to encourage the use of very large C values for the red giants.

The second possibility of obtaining larger radii appears to lie in cases in which the discontinuity in composition occurs even closer to the centre than in the cases here derived. From the behaviour of the envelope solutions at small distances

TABLE VI

Characteristics of three model stars built according to case 15 of Table V

Total mass	$10M_\odot$	$5M_\odot$	$3M_\odot$
Mass of convective core	$0{\cdot}3M_\odot$	$0{\cdot}2M_\odot$	$0{\cdot}1M_\odot$
Mass inside chemical discontinuity	$2{\cdot}0M_\odot$	$1{\cdot}0M_\odot$	$0{\cdot}6M_\odot$
Total radius	$66R_\odot$	$33R_\odot$	$20R_\odot$
Radius of convective core	$0{\cdot}3R_\odot$	$0{\cdot}2R_\odot$	$0{\cdot}1R_\odot$
Radius of chemical discontinuity	$1{\cdot}0R_\odot$	$0{\cdot}5R_\odot$	$0{\cdot}3R_\odot$
Bolometric magnitude	−5·1	−1·8	+0·8
Effective temperature	6500	4300	3100
Spectral type	F5	gK0	gM2

from the centre, it appears that such models would indeed possess very large radii. It seems uncertain however whether these models could contain a reasonable fraction of the total mass within the hydrogen-poor core and intermediate zone.

All the models discussed here, both those computed through in detail and those indicated in the above two possible extensions, involve the assumption that the change in composition occurs discontinuously. It appears necessary to investigate whether this assumption might be the cause of the limitations found in the present model. For this purpose new integrations, with continuously changing composition in part of the star, will have to be computed. Until this has been done it does not seem possible to decide whether chemical inhomogeneities alone can be responsible for the large radii of the biggest red giants.

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THE DETERMINATION OF STELLAR MAGNITUDES BY IN-FOCUS PHOTOGRAPHIC PHOTOMETRY

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Summary

The paper deals with practical points arising during the course of the Cape-Radcliffe photometric programme. The principal conclusions are :

(*a*) The errors introduced by the Schilt photometer and the process of reducing the photometer readings are considerably smaller than the total mean error.

(*b*) Tests with artificial star images show there is no detectable difference in results whether the plates are developed by brushing, rocking, or in stagnant developer. Also there seems no advantage in using backed plates for this work.

(*c*) There are considerable variations in colour equation between plate and plate, of which only a part can be accounted for by purely accidental errors. Some of the variation is correlated with exposure time. The colour equation also varies with the size of the diaphragm used in the modified Schilt photometer, an effect most probably due to the chromatic aberrations of the camera lenses. In some series of plates there is a significant change of colour equation with apparent magnitude (" photographic Purkinje effect "), and in one series this effect has probably been exaggerated by the use of plates with too long a range of spectral sensitivity for the colour correction of the lens concerned. The variation of colour equation with focal setting is small.

(*d*) The *Pg* magnitudes do not use any appreciable amount of light of shorter wave-length than 3800 A.

(*e*) The brightness of the night sky at Pretoria is about $12^{m}\cdot0$ per square minute of arc for both *Pg* and *Pv* measures; at the Cape the values are $12^{m}\cdot0$ and $11^{m}\cdot6$ respectively. Sky fog appears to have negligible effect on the current programmes. It is, however, unsafe to attempt to set up an accurate magnitude scale by in-focus photometry when within about 5^{m} of the limiting magnitude of the telescope, unless corrections can be introduced for the sky fog.

(*f*) Three methods of setting up an accurate magnitude scale by in-focus photography have been tried. Objective diaphragms proved unsatisfactory, but promising results were obtained with rotating sectors and also with a variable exposure-time method. Both procedures deserve further investigation.

(*g*) Although the Cape photometric lenses were designed for a field of diameter 15°, it is inadvisable to use them more than 3° off axis, owing to rapidly increasing distance corrections.

(*h*) For general magnitude work by any method the Earth's atmosphere sets a limit (possibly about $0^{m}\cdot01$) to the accuracy it is worth aiming at. There is a natural limit, also of about the same order, set by the inherent vagueness of the definition of a stellar magnitude.

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1. *Introduction.*—An extensive programme was started at the Cape Observatory in 1938, the primary object of which was to provide reliable *Pg* and *Pv* magnitudes* for all stars south of −30° declination whose positions and proper motions are

* We have avoided the use of the word " photographic " to describe *Pg* magnitudes, as it sometimes leads to confusion in the text, where the word is also needed in its unspecialized sense.

being determined in the photographic revision of the A.G. Zones. About 75,000 stars are included, the majority having visual magnitudes between 7 and 10. The Harvard E regions at $-45°$ were chosen for use as standard areas, as both *Pg* and *Pv* magnitudes for sequences of stars in the required magnitude range were available in *H.A.*, **89,** 1, 1932. It was quickly found, however, that these Harvard magnitudes are less accurate than the measures they were intended to calibrate, while there are serious irregular variations from region to region in both scale and zero points. The Cape programme had therefore to be extended to include the more difficult problem of establishing a satisfactory system of standard magnitudes in these regions. An account of the first attempts to establish such a system and to connect it with the North Polar Sequence by means of observations of certain clusters and of the Southern Eros Comparison Stars has already been given.* This work was subsequently rediscussed by Seares and Joyner, who deduced corrections to reduce the Cape magnitudes to the International System.†

A new chapter in the Cape magnitude programme was begun in 1944 when the Radcliffe Observatory offered its cooperation. The Cape twin photometric cameras were sent to Pretoria where they were attached to the mounting of the 74-inch reflector. After the necessary preliminary trials a regular programme was commenced in 1944 June and continued until 1947 May. The cameras were then dismounted and sent to Cambridge where they are being used to compare certain Kapteyn Areas at $+15°$ directly with the North Polar Area and thus to provide a strong connecting link between the magnitudes in the southern and northern hemispheres, some of these $+15°$ Areas having already been compared with E regions.

The original Cape magnitudes deviated from a normal Pogson scale for the fainter stars and they had to be revised as more accurate data became available. Several such revisions were made in the course of the work and finally in 1948 the whole of the available Radcliffe and Cape data were completely re-examined and fitted together to form a system of standard magnitudes. This Southern (or *S*) System is briefly described in a mimeogram which gives the magnitudes of 467 stars in the nine E regions at $-45°$. The mimeogram has been distributed to a number of observatories known to be interested in photometry and a few copies of it are still available at the Cape Observatory.

Further work on the standardization of the E regions is in progress, which will require many months before it is ready for publication. Meanwhile the purpose of this present paper is to describe the methods we have used for in-focus photometry and to discuss some of the problems of general photometric interest that have been encountered during the course of our measurements. For standard magnitudes we have relied a great deal on the Fabry method, but that will be dealt with later in a separate paper.

2. *Experimental procedure.*—All the work considered here rests on photographs taken with the Cape Photometric Cameras, at first at the Cape and later at Pretoria. A description has already been given of these cameras and of the programmes carried through at the Cape.‡ At Pretoria rather more than

* R. H. Stoy, *M.N.*, **103,** 288, 1943. R. H. Stoy and A. Menzies, *M.N.*, **104,** 298, 1944. R. H. Stoy, *M.N.*, **104,** 317, 1944.

† F. H. Seares and M. C. Joyner, *Ap. J.*, **102,** 281, 1945.

‡ *Loc. cit.*

400 pairs of plates were taken. These included plates for the following programmes :

(*a*) Zone $-35°$ to $-40°$, exposures 3 and 15 minutes.
(*b*) Zone $-52°$ to $-64°$, exposures 5 and 15 minutes.
(*c*) The Pleiades, exposures 3 minutes.
(*d*) Praesepe, exposures 15 minutes, two independent series.
(*e*) 12 Eros regions, exposures 5 and 15 minutes.
(*f*) 5 Kapteyn Areas at $+15°$, exposures 5 minutes.
(*g*) The E regions comparing E(n) with E($n-1$) and E($n-2$); two independent series of 15-minute exposures.

In (*a*) to (*f*) the E regions were used as standard fields and all comparisons were carried out with the fields at equal altitudes, with four equal exposures in the order ABBA, A being the field with the more slowly changing zenith distance. About 99 per cent of all exposures were made in a moonless sky and the remaining 1 per cent with a crescent Moon very near the horizon. The majority of the exposures were made after 22 hours S.A. standard time, as it is known that the transparency at Pretoria, at least in the dry season, often improves as the night goes on, especially during the first three or four hours after sunset. During the wet season too, when cloud formation is strongly dependent on convection, there is a tendency for maximum cloud to occur late in the afternoon, after which there may be a gradual improvement until sunrise or early morning next day. In short, the second half of the night tends to be better than the first with regard to both cloud and general transparency, although not with regard to scintillation. However, on account of the small aperture and short focal length of the cameras, scintillation rarely, if ever, has any appreciable effect on the images.

The majority of plates at Pretoria were taken without guiding and with few exceptions the definition is very good. A few plates taken at unusually large hour angles fairly early in the work were found to show elongated images. Some of the trouble was traced to a small error in the alignment of the polar axis, caused either by an ageing of the concrete piers on which the 74-inch mounting is carried, or, perhaps more likely, by a small change in the foundations after the erection of the telescope. The error was corrected, while for later work a guiding telescope was fitted to the cameras and used for all plates with fields far off the meridian, or north of the equator. All the plates with noticeably trailed images were repeated, although in no case had the trailed images on the original plates been so large as to fall outside the analysing diaphragm of the Schilt photometer.

Ilford Zenith plates, without filter, were used in the *Pg* camera; a Wratten 12 filter was used in the *Pv* camera, with Ilford Hypersensitive Panchromatic plates at the Cape, and their successors, Ilford HP2 or HP3, at Pretoria. This filter was used in contact with the photographic emulsion. The advantage of this arrangement is that the glass enclosing the gelatine filter need not be figured to any high optical standard; the disadvantages are that great uniformity of filter transmission is demanded, while the surfaces of the glass covers in the course of time tend to accumulate scratches. The transmission over the area of the filter actually used was measured directly in the Radcliffe Observatory's microphotometer and proved to be astonishingly uniform. Large-scale irregularities did not exceed $0^m{\cdot}005$; local irregularities had a maximum value of $0^m{\cdot}03$, and did not often exceed $0^m{\cdot}015$. The *Pv* magnitudes in general have errors appreciably

greater than those of the *Pg* magnitudes, but the filter itself makes no important contribution to this difference.

About half of all the plates used were backed. As described later, tests made to determine the superiority of backed plates for this particular work failed to show any detectable improvement over unbacked plates. It must not be inferred from this negative result, however, that it is desirable to dispense with backing in other branches of photographic photometry, or even in stellar photometry under different conditions.

All the plates considered here were measured at the Cape Observatory with the modified Schilt photometer, which has already been described, and which in essentials has not been altered since 1941. In 1946, however, the original Schilt method of reading the reflection of the scale in the mirror of the galvanometer, through the same eye-piece as is used for locating the image on the plate, was replaced by the more usual method of using the galvanometer mirror to throw a bright slit image on to a semi-transparent scale. The background reading can be adjusted to 100·0 by means of a variable resistance in the lamp circuit, and this is done before the reading of every star. During this adjustment the plate is trailed slowly in the neighbourhood of the pair of images to be measured, to smooth out random fluctuations of the plate grain. The adjustment of the zero for complete darkness is extremely stable, but is checked about every half-hour. The normal rate of measuring is about 75 pairs of star images per hour.

The use of the special reduction tables described in an earlier paper was abandoned in 1944. Instead, the photometer readings are reduced graphically, usually with the help of a special graphing machine. The scale of this machine is such that 1 inch represents $0^m{\cdot}5$ in one coordinate, or 10 per cent in the other, so that magnitudes can easily be read to $0^m{\cdot}01$.

The error introduced into the magnitude by the photometer itself (as distinct from the photographic emulsion, the sky, etc.) depends on several factors, such as the size of the diaphragm, the nature of the plate grain and the structure of the star images. We will call it ϵ_m. For plates taken with the Cape photometric cameras the standard diaphragm used in the photometer has a projected diameter on the plate of 0·12 mm. With this diaphragm an important contribution to ϵ_m comes from uncertainty of the background reading, due to random fluctuations in the plate grain. To evaluate the size of this error, repeated background readings on roughly the same area were made on a number of plates. The average deviation from the mean when static readings were taken was 1·45 per cent, but only 0·48 per cent when readings were made by the method normally used, viz. with the plate being slowly trailed. For a larger diaphragm or with finer grain plates these fluctuations would naturally be smaller. This uncertainty in the background setting does not affect all images equally; its effect is directly proportional to the photometer reading and is therefore more serious for the fainter than for the stronger images.

The other important contribution to ϵ_m comes from the centring of the image in the diaphragm. This error appears to depend on the size of the image, tending to be larger for the stronger images than for the weaker. Repeated settings on a number of star images of different size indicated that the average deviation from this source is 0·27 per cent. Thus for a single average image giving a reading of 50 per cent, the average ϵ_m from these two causes will be $\sqrt{(0{\cdot}24^2+0{\cdot}27^2)}$, or 0·36 per cent. The normal unit of measurement is a pair of images, but the

background is read only once for such a pair. The random fluctuations of such a pair will be given by $\sqrt{(0{\cdot}24^2 + \frac{1}{2}0{\cdot}27^2)}$, or 0·31 per cent. This corresponds to a standard deviation 0·39 per cent, when only ϵ_m is taken into account, and this with an average reduction curve is equivalent to $0^m{\cdot}016$.

From plates that have been reduced in duplicate it appears that the errors introduced during the graphical reduction are of the same order of magnitude as ϵ_m. As the standard deviation of a magnitude determined from a normal programme plate is $0^m{\cdot}05$, taking all errors into account, it will be seen that only a minor part of the total error arises in the process of measurement and reduction. The increase in accuracy which would result from measuring and reducing plates in duplicate would be small and certainly not worth the labour involved.

Early, during the measurement of the Pretoria *Pv* plates, it was noticed that the differences between the photometer readings for a pair of images was greater than it had been for plates taken at the Cape. In extreme cases one of the images would be strong, while the other was very faint or absent altogether. The measurers (M. and S.) were of the opinion that this was due to local irregularities in the sensitivity of the emulsion. R., however, suggested that the trouble might be attributed either to scratches on the filter, or possibly to the method of development, but a further examination of the evidence did not favour either of these ideas. Were the irregularity due to marks or scratches on the filter, some at least of the affected images should show signs of distortion or partial eclipse. There are such images, but only at the rate of about 1 in 10,000, while abnormally irregular images in that particular batch of plates occur at a rate nearer to 1 in 200. The evidence at present favours the idea that irregular emulsion sensitivity is responsible, but it was the occurrence of these abnormally weakened images that led to the tests of development described in the next section.

3. *Development of plates.*—At the Cape all plates had been developed in a tank, in a fairly weak amidol developer. At Pretoria they were all brush developed in a strong *M-Q* developer, being first stored for ten hours or more after the last exposure, in order to minimize the effects of possible progressive changes in the latent image. (The effect of such changes would in any case be considerably reduced by the ABBA ordering of the exposures.)

While the advantages of brush development in most photographic photometry are undisputed, we have been unable to establish that it shows any superiority for the photometry of in-focus star images. Two series of special exposures with artificial star images were made at Pretoria on HP2 or HP3 plates and these were measured at the Cape in the same way as the regular star plates. In the first series six plates were used, three backed, three unbacked, each containing 320 exposures in groups of four, with exposure times 1, 2, 4 or 8, the unit being about $\frac{2}{3}$ of a second. These were developed in pairs, each consisting of one backed and one unbacked plate, in brushed, rocked or stagnant developer. The exposures unfortunately showed a systematic progression from beginning to end of each plate (due mainly to the pendulum shutter used), and the reduction of the measures had to be arranged to eliminate this as far as possible. Further, the residuals showed a correlation with the image strength, so that they had to be reduced empirically to a standard photometer reading of 40 per cent. The mean deviation of an individual image, expressed as a percentage of the full-scale reading, was as follows: brushed 0·80, rocked 0·86, stagnant 0·86. These residuals would, on an ordinary star plate, be roughly equivalent to $0^m{\cdot}03$.

Since the reduction of this first series had been slightly unsatisfactory, for the reasons already given, a second series of tests was made with more elaborate arrangements to ensure uniformity of exposure. There were nine plates in this series and the artificial star images came in groups of nine, with at least 25 groups per plate. The average number of images measured per plate was 78 and the mean residuals per image are given in Table I. It is not known why the HP3 rocked plate shows unusually small residuals, although it may be noted that it also has the least number of measured images.

TABLE I

Errors in artificial star images as a function of development

	Brushed			Rocked			Stagnant		
	(*a*)	(*b*)	(*c*)	(*a*)	(*b*)	(*c*)	(*a*)	(*b*)	(*c*)
	per cent			per cent			per cent		
HP2	74·7	1·36	30	73·9	1·25	30	75·3	1·16	30
	60·1	0·99	30	58·3	1·10	30	61·4	1·00	30
	54·2	0·99	30	52·2	1·44	30	54·7	0·98	30
HP2	77·0	1·38	25	78·6	1·38	26	78·8	0·77	25
	61·5	1·46	25	62·0	1·28	26	64·2	1·03	25
	53·0	1·32	25	53·9	1·41	26	56·2	1·10	25
HP3	74·5	1·21	24	74·1	0·49	18	76·2	0·92	25
	58·2	0·92	24	56·2	0·60	18	61·3	0·66	25
	46·5	0·99	24	43·2	0·97	18	49·3	0·90	25

(*a*) mean reading; (*b*) mean residual; (*c*) number of images measured.

The mean residual from all measured images, all plates, is 1·08 per cent, which corresponds to a standard deviation of ± 1·35 per cent, and this on an ordinary star plate would be equivalent to about $\pm 0^{m}\cdot 05$. Judging by the first tests and by our experience with star measures in general, for a purely photographic error this is rather larger than usual, but the second series confirms the result from the first, that there is no detectable advantage in brushing and, indeed, stagnant developer seems as good as any.* However, as the photometric programme had already proceeded rather far at the time of these tests, it was felt that for the sake of uniformity the original method of development with brushing should be continued. In any fresh programme it might be worth while to use stagnant developer, provided it be remembered that a lack of agitation in the developer is not synonymous with a lack of all other precautions. For example, it is desirable to keep the plate horizontal, and essential to start by pouring developer on to the plate, and not by putting the plate into the developer.

A tentative explanation of the reason why brushing is in this case of no advantage may be offered. Even in the most thickly populated star field the images on the plate have to be fairly well separated (the Schilt method of measurement would not be satisfactory otherwise) and it is easy to see that only an extremely small fraction of the total amount of silver halide in the plate is reduced to metallic silver. Even taking into account the background fog (which is small), the developer has to do much less work per unit area of plate than in almost any other branch of photometry; the demands upon it are localized to relatively widely

* *Cf.* F. E. Ross, *Ap. J.*, **84**, 256, 1936.

scattered points so that the ordinary processes of diffusion are sufficient to replenish the developer in the vicinity of each image. Further, as far as systematic errors are concerned, any developer effect, such as the Eberhard effect, which modifies equal images equally, is of no importance since the photometric scale is obtained from a selection of the images themselves.

In both series of development tests an additional row of very strong images was put close by the last row of regular images. Any effect on the latter appeared to be not only very small, but no greater with the unbacked than with the backed plates. Thus it is doubtful whether backed plates have any advantages over unbacked for the photometry of normally exposed in-focus star images. Again, this conclusion is not to be generalized or applied to any other branch of photographic photometry.

4. *Colour corrections.*—In general the natural colour system of any set of photometric observations will differ from that adopted for standard. In some forms of photometry it may be possible to compute the necessary colour corrections, by consideration of the spectral sensitivity of the receiver, the optical properties of the telescope, etc., but this is certainly not the case for in-focus plates taken with a refractor. The colour corrections for these plates depend not only on such calculable factors, but also on the accidental circumstances at the time of observation and on the way in which the individual plates are processed and measured. In the last analysis the colour corrections have to be obtained in a purely empirical way by observation of standard stars of known colours. The reduction of some 300 pairs of plates taken at Pretoria and the Cape to a single colour system—that of the *S* magnitudes for the E regions—allows examination of the way in which the colour correction varies from plate to plate. In the following paragraphs we give an account of the results of such an examination and of how far these results can be explained qualitatively by the colour characteristics of the apparatus.

4.1. *Random fluctuations of colour equation.*—At first sight the most striking fact about the required colour corrections is the size of the random fluctuations from plate to plate, even in a programme where observing, processing and measuring conditions have been held as uniform as possible, and where colour characteristics would be expected to be nearly constant.

If for the moment we confine our attention to the 131 pairs of plates covering the $-35°$ to $-40°$ zone, which were taken, processed and reduced under relatively homogeneous conditions, and consider the differences between plates with the same length of exposure, taken on the same night, we find the results set out in Table II. Two plates with the same exposure, taken on one night, give one difference, three plates give three, and so on.

These fluctuations are in part accidental. There are about 36 standard stars per plate, and these have to define the colour characteristics of that plate over a range of colour index about $-0{\cdot}2$ to $+1{\cdot}8$. Even if the standard magnitudes and standard colour indices were perfectly accurate, the measured magnitude for each standard star on any one particular plate would have a mean square error about $0^{m}{\cdot}05$, so that the correction of the plate to the standard colour system, usually supposed to follow a linear law with regard to colour index, is necessarily subject to some accidental error. We cannot in general predict the error accurately, since the distribution of standard stars with regard to colour index is irregular and varies from E region to E region, but, assuming a Gaussian distribution of errors, it is possible to estimate it roughly in two ways as follows.

TABLE II

Variation in the colour equation for plates taken under similar conditions

Difference in *C.E.*	Number of plates			
	Pg		*Pv*	
	3 min.	15 min.	3 min.	15 min.
0·00 × *C.I.*	10	11	7	4
0·01	13	8	10	4
0·02	9	2	15	2
0·03	13	4	12	6
0·04	6	1	4	1
0·05	2	1	5	6
0·06	5	1	4	1
0·07	3	1	1	1
0·08	...	...	3	1
0·09	1	...	...	
0·10	...	...	...	...
0·11	...	...	...	...
0·12	...	...	...	1
Mean square difference	0·034 × *C.I.*	0·025 × *C.I.*	0·035 × *C.I.*	0·044 × *C.I.*

First, suppose 36 standard stars in two equal groups, of colour indices 0·3 and 1·3 respectively. The colour correction is then a straight line drawn through two points, each of whose coordinates has a standard error 0·012. The slope of the straight line has a m.s.e. 0·017. The difference in colour correction between two plates has a m.s.e. 0·024, rather more than half of the amount observed.

Secondly, we have taken an actual E region selected at random, viz. E3, and with the aid of a table of the error function plus a list of random numbers*, taken from 0 to 999, an experimental Gaussian distribution of errors (m.s.e. 0·05) was drawn up and applied to the standard magnitudes of the region ten times over. This gave ten lists of synthetic errors which could be plotted against the standard colour indices, and a straight line drawn through each plot. The slopes of these lines, in other words the synthetic colour equations for ten cases, when taken in pairs with each other, gave an average difference (without regard to sign) 0·02. (The same technique applied to another area gave 0·03.)

The two estimates of the accidental error in the determination of a colour equation are in fair agreement and indicate that rather more than half of the observed random fluctuations can be so explained. We believe, however, that the remaining part of the observed fluctuation is to be attributed to real physical variations between plate and plate, for it will be noticed that the distribution of the observed variations is not Gaussian. This is particularly the case for the 3-minute plates. Moreover, experience has shown that the application of the colour equation derived from the E region stars has improved the agreement of the magnitudes deduced for the stars in the other region photographed on the same plate.

* R. A. Fisher and F. Yates, *Statistical Tables for Biological, Agricultural and Medical Research*, 1948.

The physical reason for any real variation in colour equation for plates taken under apparently similar conditions is still not entirely clear to us. There may be a combination of several factors involved, some of which are discussed below. We believe, however, that the cause should be sought in a variation of structure of the individual star images, rather than in the sky transmission, or the ordinary colour sensitivity of the plate, or in changes of the colour correction of the lenses. We have failed to find any marked correlation between the variation in colour equation of *Pg* and *Pv* plates which were exposed simultaneously, so that the sky probably makes a relatively small contribution. Plates of the same brand are found to show much smaller fluctuations in colour sensitivity when used for observations by the Fabry method, where an extended image of uniform density is measured. Again, it has been suggested that significant changes might be caused by differential temperature effects between the components of each camera lens, but we have been unable to test this hypothesis. A possible correlation between colour equation and the seasonal temperature was examined, but all that has been found is a range of variation from midsummer to midwinter of about 0·01 *C.I.* for *Pg* and 0·015 for *Pv*.

4.2. *Effect of exposure time.*—There is a systematic difference between the colour corrections required for the 3- and 15-minute *Pg* plates respectively. Of the plates covering the $-35°$ to $-40°$ zone, 66 pairs received 15-minute exposures and 65 pairs 3-minute exposures; otherwise all the plates received similar treatment. This systematic difference is fairly consistent and amounts to 0·03 *C.I.*, in the sense that the longer-exposed plate appears to be more sensitive to shorter wave-lengths. The mean differences of magnitude for stars giving equal photometer readings on the 15- and 3-minute plates respectively is $1^{m}·6$.

Table III shows the mean colour corrections required by the plates to reduce them to the *S* system, tabulated according to the E region with which the plate was standardized. Each entry is the mean for about seven plates. It will be noticed that the *Pv* plates (HP2) do not show a systematic difference between the 3 and 15-minute exposures.

TABLE III

Mean colour corrections for the $-35°$ to $-40°$ zone

E Region	*Pg*			*Pv*		
	3 min.	15 min.	3–15 min.	3 min.	15 min.	3–15 min.
1	0·10 × *C.I.*	0·05 × *C.I.*	+0·05	0·04 × *C.I.*	0·03 × *C.I.*	+0·01
2	0·06	0·03	+0·03	0·07	0·05	+0·02
3	0·07	0·03	+0·04	0·05	0·07	−0·02
4	0·04	0·02	+0·02	0·06	0·05	+0·01
5	0·06	0·05	+0·01	0·04	0·03	+0·01
6	0·05	0·03	+0·02	0·02	0·05	−0·03
7	0·05	0·04	+0·01	0·03	0·02	+0·01
8	0·07	0·03	+0·04	0·04	0·04	+0·00
9	0·06	0·03	+0·03	0·02	0·04	−0·02
All	0·06	0·03	+0·03	0·04	0·04	0·00

4.3. *Effect of the size of the measuring diaphragm.*—Apart from short special investigations such as are mentioned below, all our measuring has been done with a diaphragm of diameter 0·12 mm., as projected on the plate. There is no difficulty in changing the diaphragm on the photometer, but our experience so far has led us to prefer one standard diaphragm, chosen carefully in relation to the average size of the star images in order to give the greatest photometer accuracy. Too small a diaphragm exaggerates the effect of background irregularities and reduces the sensitivity for strong images; too large reduces the sensitivity for the weaker images and also hinders the separation of nearby stars. (On this last point we do not advocate the de-centring of images within a large diaphragm to avoid close companions, as the photometer reading is liable to be altered thereby, especially with stronger images.) We find that with the appropriate diaphragm the photometer works best with images from one to four magnitudes brighter than the faintest discernible on the plate.

Typical image diameters have been measured by Mr Lourens with the Cape Hilger measuring machine and are given in Table IV. These diameters may be taken to refer to something slightly larger than the "burnt-out" core of the image. It may be noted that the difference in brightness between two stars for which photometer readings are 80 and 20 per cent respectively is about $2^{m}\cdot8$, so that the rate of growth of image diameter is 0·02 mm. per magnitude at about three magnitudes above the plate limit.

TABLE IV

Approximate image diameters in microns

Photometer reading	*Pg*		*Pv*	
(per cent)	3 min.	15 min.	3 min.	15 min.
80	59	57	66	67
50	87	85	93	93
20	118	113	120	118

We are here chiefly concerned with the effect of diaphragm diameter on the colour characteristics of our measures. Some of the brighter stars on the Pretoria plates of the Eros regions were measured both with the standard diaphragm and with another of twice this diameter. With the larger diaphragm blue stars on the *Pg* plates appear relatively brighter than they do with the normal diaphragm, the difference being about 0·085 *C.I.* Similar measures on the corresponding *Pv* (HP3) plates do not show this effect.

Plates of the first Pretoria series of E region intercomparisons were also measured with both diaphragms, in an attempt to extend the measured range of magnitude. The measures with the larger diaphragm were confined almost entirely to stars which had been deemed too bright to measure with the normal diaphragm. The colour corrections needed to reduce the measured magnitudes to the *S* system differ by 0·085 *C.I.* for the *Pg* plates and by 0·09 *C.I.* for the *Pv* (HP2) plates, in both cases in the sense that blue stars appear relatively brighter when measured with the larger diaphragm. Here, however, it is not possible to say that the observed change of colour characteristic is purely the result of the method of measurement (as it was for the Eros plates), since the same set of stars was not measured with both diaphragms. The average magnitude difference between the bright stars measured with the large diaphragm and the fainter stars measured with the normal diaphragm was $2^{m}\cdot2$ for the *Pg* plates and $1^{m}\cdot5$ for *Pv*.

4.4. *Variation of colour correction with magnitude.*—Some of the apparent difference in colour sensitivity for the bright and faint stars in the last example may possibly be due to a photographic effect analogous to the Purkinje effect in visual photometry, i. e. to a variation of colour characteristics with the brightness of the stars concerned. When investigating systematic differences between two sets of magnitudes, it is our regular practice to arrange the differences for individual stars in a rectangular array, the columns of which contain half-magnitude intervals and the eight rows approximately equal ranges of colour. This detects any variation of the relative colour correction with magnitude. In a few cases the variation has been quite conspicuous. An outstanding example is a set of *Pv* magnitudes of E region stars, derived from 3-minute exposures made at the Cape between 1939 and 1941. The difference between these magnitudes and those derived from plates taken at Pretoria with the same camera, but having 15-minute exposures on HP2 emulsion, can be well represented for the range of magnitude in common by

$$\text{Cape } Pv - \text{Pretoria } Pv = -0{\cdot}060\,(10{\cdot}4 - m)\,(C.I. -0{\cdot}40), \quad 7{\cdot}0 < m < 9{\cdot}0.$$

The Pretoria magnitudes had themselves been compared with a series derived by the Fabry method, and relative to them had shown no variation of colour characteristic with magnitude. The colour characteristics of the Fabry measures should be completely independent of magnitude, since they depend on extended images of nearly constant density. It can be concluded, therefore, that the Pretoria *Pv* observations are also free from a systematic variation in their colour characteristics with magnitude, but that this particular series of Cape observations is not.

The corresponding series of Cape *Pg* magnitudes showed no such effect in the range $8^{m}{\cdot}0$–$9^{m}{\cdot}0$ when compared with the Pretoria *Pg* values, but did so when compared with Fabry measures made by Mr Cousins in the range $6^{m}{\cdot}5$–$8^{m}{\cdot}5$. After each series of magnitudes had been reduced as nearly as possible to the standard colour system by the application of simple linear corrections, the difference between them could be well represented by

$$\text{Cape } Pg - \text{Fabry } Pg = 0{\cdot}045\,(8 - m)\,(C.I. -0{\cdot}10), \quad 6{\cdot}5 < m < 8{\cdot}5.$$

It cannot be deduced with certainty that the Pretoria *Pg* measures would show a similar change of colour characteristic with magnitude when compared with magnitudes determined by the Fabry method, as the ranges of magnitude in the comparisons quoted above are different. The observed difference in colour between the 15-minute and 3-minute Pretoria *Pg* plates in the $-35°$ to $-40°$ zone suggests that the effect may possibly be present, but that, if it can be represented by a formula similar to that for the Cape plates, the coefficient will be only 0·02. The necessary Fabry observations to check the colour characteristics of the Pretoria *Pg* magnitudes are now being made.

The existence of a "photographic Purkinje effect" in various series of magnitudes has long been recognized*, but its practical implications have usually been ignored. Its effect must be removed from the measured magnitudes before these can be used for any satisfactory statistical discussion. The size of the corrections needed can be kept to a minimum by the choice of suitable plates for the spectral region being observed and by limiting the range in magnitude of the stars measured on any plate.

* See, for example, F. E. Ross, *The Physics of the Developed Photographic Image*, chapter III, 1924.

5. *Colour properties of the apparatus.*—To some extent the observed colour behaviour of our magnitudes can be explained in terms of the colour properties of the apparatus employed, or of the conditions at the time of observation. We shall discuss these under the following heads :—(1) plate sensitivity, (2) the chromatic aberrations of the lenses, (3) the effects of errors in the focal setting, (4) the effect of differential extinction.

5.1. *Plate colour sensitivity.*—Some of the differences between the *Pv* observations made at the Cape and those made at Pretoria may be attributed to differences in the plates employed. Ilford Hypersensitive Panchromatic plates were used for the Cape observations, but were unobtainable at the time the Pretoria observations were made. Ilford HP2, and later HP3, plates were therefore chosen as being their nearest equivalents. These three emulsions differ noticeably in their red sensitivity. The actual ratio of the sensitivities probably depends on experimental conditions, but the following ratios, obtained with exposures of 16 seconds, give a rough indication of the relative speeds at various wave-lengths of the two types of plates used at Pretoria. The measures assume equal speed at 5200 A., which is not far from the truth.

TABLE V

Relative colour sensitivity of HP2 and HP3 plates

Wave-length (A.)	5200	5400	5600	5800	6000	6200	6400	6500
HP2/HP3	1·0	1·3	1·5	1·5	1·5	2·2	3·0	17·5

The HP3 plate is sensitive to about 100 A. further into the red than is the HP2. Under constant conditions a change from one of these emulsions to the other in our work altered the colour correction to be applied to the measures by about 0·14 *C.I.*

Unfortunately similar measures of the relative sensitivity of the Hypersensitive Panchromatic plates used at the Cape between 1938 and 1941 were not made. From the spectral sensitivity curve in *Lund Annals*, **7**, 52, 1938, and from the spectrogram given in the Ilford pamphlet *Photography as an Aid to Scientific Work*, we are led to believe that these plates were sensitive to well beyond 6600 A.; in fact Stoy used them in 1935 to observe nebular lines as far into the red as 7136 A.*

There may have been slight modifications, especially near the long wave-length cut-off, in the Ilford Zenith plates used for the *Pg* measures during the eight or nine years covered by the Cape and Pretoria observations. In tests made in 1943 and described in the following section, the plates proved to be decidedly orthochromatic, the cut-off being at about 5900 A., although the sensitivity was very low above 5100 A., where the cut-off is usually supposed to be. The average difference in the colour correction required by the Cape *Pg* plates and the earlier Pretoria plates having the same exposure time was 0·08 *C.I.* in the sense that blue stars appear relatively brighter on the Pretoria plates. A part of this change is certainly due to the difference in altitude above sea-level (Cape 8 m., Pretoria 1542 m.). But there is also a change between earlier and later Pretoria plates of about 0·04 *C.I.*, in the same sense.

5.2. *Chromatic aberrations of the lenses.*—The colour curves of the two objectives were determined by placing a large 45° prism in front of each lens in turn, and using it to obtain a series of spectra of Sirius, each spectrum being obtained with a different focal setting. The curve for the *Pg* lens is practically

* R. H. Stoy, *Lick Obs. Bull.*, **17**, No. 480, 1935.

flat between 4350 A. and 5500 A., but rises steeply in the violet. The curve for the *Pv* lens is very similar, but the minimum is further to the red; it is very nearly flat between 4500 and 5600 A.

TABLE VI

Colour curves of the camera objectives

(Distance from minimum focus, in millimetres)

Wave-length	*Pg*	*Pv*
3800	3·63	3·91
4000	1·28	2·06
4200	0·46	0·81
4600	0·00	0·14
5000	0·00	0·00
5400	0·14	0·00
5800	0·57	0·46
6200	1·03	0·96
6600	1·60	1·46

In practice neither lens is set at the minimum focus, but at the best compromise focus for the range of wave-lengths to which the plate (or plate-filter combination) being used is sensitive. For the *Pg* lens and Zenith plates this focus is about that for 4300 A., and all wave-lengths between the red limit of plate sensitivity and 4050 A. can be considered as sharply in focus. The light to the violet of 4050 A. is slightly out of focus and tends to increase the size of the star image. This increase of size will become more marked with increasing exposure and will, of course, be greatest for the blue stars. This may be the reason why the *Pg* plates appear to be of increased blue sensitivity for the longer exposures, or for measures made with a large diaphragm. Incidentally, light of shorter wave-length than 3800 A. is so far out of focus that it makes virtually no contribution to the star image. Magnitudes found with this lens can therefore be expected to be relatively insensitive to the effects of the Balmer lines near the series limit, or of the Balmer continuous absorption.

For the *Pv* lens used with a panchromatic plate and a Wratten 12 filter the focal setting will be about that for 5600 A. All wave-lengths between 5050 A. (the practical limit of transmission by the filter) and 6150 A. will then be practically in focus. Light of longer wave-length than 6150 A., up to the limit of the plate, will be slightly out of focus, and the redder the star or the greater the red sensitivity of the plate, the greater the effect of this out-of-focus light will be. On the other hand, the colour curve of the lens is not nearly so steep in the red as it is in the blue, so that the effect of the out-of-focus red light on the *Pv* magnitudes will not be so strong as that of the out-of-focus blue light on the *Pg* magnitudes. We can expect the effect to become more conspicuous, however, if the sensitivity of the plate extends much into the red beyond 6500 A., and it is possible that this is the reason why the series of *Pv* plates taken at the Cape show a marked variation of colour characteristics with the brightness of the image, the variation being in the sense that observations with these plates make bright red stars appear redder than faint red stars of the same intrinsic colour.

5.3. *Effect of errors in the focal setting.*—The lenses and plate holders were carefully squared off with respect to each other when the cameras were attached to the mounting of the 74-inch reflector and the focus was set and left unaltered throughout the whole of the photography of the $-35°$ to $-40°$ zone. A brief

series of tests was then made to examine the effect of deliberate changes of focal setting on the measured magnitudes. The results were quantitatively somewhat irregular, but qualitatively they indicated that the measured brightness of stars increases as the plate is shifted away from the optimum focal setting and that this increase is most marked for the brightest stars. These results are exactly what might be anticipated. When the optical image is focused sharply, the photographic image does not spread appreciably until the centre is heavily over-exposed, and thus with well-exposed stars any change which distributes the same amount of light over a greater area of the plate tends at first to darken the outer parts of the image, with relatively little diminution of the photographic density at the centre. The average change of brightness found was about $0^{m}\cdot3$ per 1 mm. change in focus (the lenses working at $f/10$).

A more important result was that there was no evidence of a change in colour equation exceeding $0\cdot06 \times C.I.$ per 1 mm. change in focus. In other words, the small deviations from the optimum focal settings which may have occurred in practice, due chiefly to seasonal temperature changes, are unlikely to have produced any appreciable change in the colour characteristics of the magnitudes. At the level at which the cameras were used, the annual temperature range in the 74-inch turret is about 21 deg. C. and the diurnal range about 4 deg. C. Changes during the one and a quarter hours required to get a "15-minute" plate will clearly be negligible. The seasonal changes will affect standard stars equally with those being measured and hence should be automatically eliminated in the plate reductions.

5.4. *The effect of differential extinction.*—No rigorous examination has been made of the extinction at either the Cape or Pretoria, but it is known to vary considerably with time (as it probably does at most other places in the world), so that mean values, however carefully determined, would not be of much practical use, and might even prove misleading. At Pretoria the transparency is at its best during and immediately after the rainy season but, as might be expected, there is a gradual deterioration as the dry season advances. In calm weather this is often complicated by the nocturnal change already mentioned, following the settling down of dust and smoke during the night, after the convection stirred up by the warmth of the Sun has died away. The increase of transparency is generally most noticeable during the early hours of darkness, but may continue all night. On at least two occasions stars have been followed for about four hours, from the vicinity of the zenith, $\sec z = 1\cdot0$, to about $\sec z = 1\cdot6$, and at the end of the run have shown practically the same apparent *Pg* brightness as at the beginning, the zenith extinction diminishing perhaps from $0^{m}\cdot5$ to $0^{m}\cdot3$ during this time. These were extreme cases and usually such nights can be picked out by the naked eye, but important variations may occur on an average night, and may completely elude even an experienced observer, unless he is making special measures of the extinction. It is necessary to construct photometric programmes with this possibility in mind.

For the present programme, systematic changes of extinction, provided they occur fairly smoothly and not too rapidly, should have little effect on the measured magnitudes owing to the ABBA order of exposing and to the fact that the four exposures are spaced systematically about the sidereal time at which the zenith distances of the A and B regions are equal. The normal tolerance allowed here was about 5 minutes, and where this was exceeded the values of $\sec z$ were usually

checked to ensure that their deviations from the optimum values were of no importance. Differential extinction corrections were applied only to those plates which were taken at zenith distances greater than 40°, a zenith extinction of $0^m.32$ *Pg* and $0^m.20$ *Pv* being assumed. These will be under-corrections on all but the best nights.

It may be anticipated that red stars will show smaller extinction than blue; indeed, such an effect of differential extinction for the Cape *Pg* magnitudes was noted in a previous paper. The magnitude of the effect found there amounted to $0^m.04$ for a unit difference of colour index and a change in sec z of 0·4. Brief tests of this differential extinction as between red and blue stars were made at Pretoria. While not exhaustive, they suggest that the effect is negligible on most nights for sec $z < 1.3$, but that, for a unit difference of colour index, it might amount to $0^m.03$ or $0^m.04$ at sec $z = 1.6$. This is of the same order as the effect found at the Cape. No corrections have been applied for this effect.

Variations between night and night due to smoke, etc. may also be expected to affect blue stars more than red, but experience has shown that changes in the selective extinction are usually small compared with the changes that take place in the total extinction. A small amount of the observed variations in colour characteristics from plate to plate may be due to this cause, but the use of sufficient standard stars of a good range of colour to determine the colour correction for every plate should ensure that there is no appreciable effect on the magnitudes finally produced.

6. *The effect of sky fog.*—Measures made in connection with the Fabry observations indicate that the brightness of the night sky at Pretoria is about $12^m.0$ per square minute of arc for both *Pg* and *Pv* measures. Similar measures made by Cousins at the Cape give $12^m.0$ per square minute of arc for *Pg* and $11^m.6$ for *Pv*. These figures refer to the sky at the zenith compared with a star at the zenith whose brightness has not been corrected for atmospheric extinction.

The exact value of the sky brightness is not needed for the ordinary zone programme, but it is important to remember that every star image results from one exposure to star plus sky, with three other exposures to sky alone. The intensity of the natural light of the sky was probably in effect somewhat increased by false light arising in the cameras themselves. The lenses each have eight air-glass surfaces and the polish on some of these surfaces had somewhat deteriorated prior to their use at Pretoria, probably as a result of dewing at the Cape, where the Observatory is very near to the sea. As used in the 74-inch turret, the lenses were exposed to very little sky outside the area actually photographed. During the second half of the programme the diffused false light was still further reduced by a diaphragm fitted about halfway down each camera to cut out the light reflected from the sides of the box. The lenses have since been repolished and bloomed by the makers.

An attempt was made to evaluate the effect of this scattered light by examining the individual images of ten relatively bright stars and ten other stars approximately two magnitudes fainter on a series of ten pairs of plates, each of which had had four exposures of fifteen minutes. The average arithmetical difference between exposures one and four was larger than that between exposures two and three ($0^m.04$ *Pg*, $0^m.03$ *Pv* as against $0^m.03$ *Pg*, $0^m.02$ *Pv*), but the systematic differences were equally small for both ($0^m.016$ *Pg*, $0^m.018$ *Pv*). The systematic difference between the bright and the faint stars (i. e. the change of scale between the two

exposures) averaged $0^m{\cdot}02$ for both the *Pg* and the *Pv* plates, but was 60 per cent smaller for exposures two and three than for exposures one and four.

Included amongst the ten pairs of plates examined are three pairs for which the systematic differences between the exposures were considerably larger than the average. The explanation of these outstanding plates is probably atmospheric. Experience in this and other branches of photometry shows that even on nights which appear good to the naked eye, changes in transparency of the order of $0^m{\cdot}05$ or more are not uncommon.

It would appear, then, that any systematic effect of the light of the night sky on the measured magnitudes is quite small and is, in any case, minimized by the ABBA ordering of the exposures. In the case of the two series of intercomparisons of the E regions made at Pretoria, the order of the exposures was deliberately inverted from the first to the second series. Should there be an appreciably systematic error arising from the light of the sky, and depending on the order of exposure, it should show as a systematic difference between the results of these two series. No clear indication of such a systematic difference was found.

Though it may be quite safe to ignore the effects of the light of the night sky when using a camera to compare two regions of the heavens, it does not follow that it is equally negligible when the same camera is being used to set up a magnitude scale. In fact it is of practical importance to estimate as carefully as possible the effect of the night-sky light in order to get some indication of the point at which neglect of it is likely to introduce appreciable errors into determinations of scale. Though the sky light may be below the threshold value of the plate as a whole, it will, when added to the light of a star, both increase the blackness of the main body of the star image, and slightly enlarge its diameter. When several exposures are made on the same plate, every portion of the plate is exposed to the sky for the whole period, and owing to the peculiarities of the photographic emulsion the photographic effect of the sky light may be different, according to whether exposure to it precedes, coincides with, or follows that to the star.* In what follows, this complication is ignored.

For the Cape Schilt photometer, the sizes of the star images measured with any diaphragm range from the size of the diaphragm itself (for the brightest measurable stars) to about one-quarter of this area for the faintest. For the bright stars the centres of the images are "burnt out" and the sky contribution to the outer part only of the images will be of any importance. For the faintest stars, the sky light will contribute only to the portion of the plate covered by the star image, i. e. over an area about one-quarter that of the diaphragm. Thus as far as order of magnitude is concerned, we may not be far wrong in assuming that the sky effect is equivalent to adding the light of n stars of magnitude m_s to every star observed, where m_s is the magnitude of the light coming from an area of night sky equal to one-quarter that of the analysing diaphragm and n is the number of exposures made on the plate. We take no account of the reciprocity failure of the emulsion or other photographic phenomena making the effect of repeated exposures not entirely additive.

Suppose now we try to set up a scale by observing two stars of magnitude m_1 and m_2 and ignore the effect of the sky light. The observed magnitude difference will not be $m_2 - m_1$, but $m_2 - \Delta m_2 - (m_1 - \Delta m_1)$, where $m - \Delta m$ denotes the magnitude of the combined light of star and sky. The value of Δm corresponding

* G. Alter, D. R. Barber and D. L. Edwards, *M.N.*, **100**, 529, 1940.

to a range of values of $m_s - m$ is given in the following table, which shows that if the effect of sky be ignored, there is likely to be an appreciable error in the deduced scale unless the faintest star observed is at least five magnitudes brighter than m_s. (We note in passing that m_s approximately equals the magnitude of the faintest star detectable with the camera with an indefinitely prolonged exposure.)

TABLE VII

The effect of sky fog on determinations of scale

$m_s - m$	0	1	2	3	4	5	6
Δm	0·753	0·364	0·160	0·066	0·027	0·011	0·004

The angular area corresponding to the standard 0·12 mm. diaphragm is 0·16 square minutes of arc. As the brightness of the sky is about 12^m per square minute of arc, we deduce that m_s is about $15^m{\cdot}5$ per single exposure, or about $14^m{\cdot}0$ for four exposures. The cameras therefore should not be used for setting up scales of magnitude much fainter than $9^m{\cdot}0$. For stars between $9^m{\cdot}5$ and $10^m{\cdot}0$ we might fear an error scale of about 2 per cent, and between $10^m{\cdot}0$ and $10^m{\cdot}5$ about 3 per cent. The argument applies only to scale determinations and does not apply to ordinary magnitude work, done by intercomparison with a standard area.

Neglecting scintillation, the intensity of an in-focus star image is nearly proportional to the area of the telescope objective, whereas the effect of the light of the sky is proportional to the solid angle subtended by the objective at the focus, i. e. to the inverse square of the aperture ratio. For stars of a given magnitude and a telescope of given aperture the effect of the light of the sky will therefore diminish as the focal length of the telescope is increased, until the size of the image begins to be chiefly determined by scintillation.

Consider, for example, the case of the Victoria Refractor at the Cape, of which the aperture is 61 cm. and the focal length 686 cm. For plates taken with this telescope an analysing diaphragm having a projected diameter of 0·24 mm. is used. The corresponding value of m_s is $18^m{\cdot}3$ per exposure, so that it would be safe to use this telescope to set up an accurate magnitude scale by simple in-focus methods as far as the twelfth magnitude.

It should be stressed that these estimates are comparatively rough, and take no account of the known peculiarities of photographic emulsions with regard to reciprocity failure. They serve chiefly as a warning.

7. *Setting up a magnitude scale.*—The scale of the *S* magnitudes is that defined by Fabry observations made at Pretoria and at the Cape. Before the Fabry method was adopted as the most reliable for setting up a magnitude scale, three other ways were tried, each using in-focus photometry, (*a*) with rotating sectors, (*b*) by variable exposure times, (*c*) using diaphragms. The experiments with method (*c*) were quickly abandoned, as the sector diaphragms employed altered the structure of the star images. The experiments with the two other methods were rather more extended and are briefly described in the paragraphs that follow.

7.1. *In-focus photography with rotating sectors.*—This method of setting up a magnitude scale was tried with the Cape *Pg* camera. The standard plate holder was replaced by an adaptor carrying a smaller plate and equipped with a high-speed electric motor driving one of a series of interchangeable sectors of 8 cm. diameter, which cut down the incident light by approximately 0·5, 1, 1·5, 2 or

2·5 magnitudes. The motor was situated behind the plate, its shaft lying as closely as possible to the plate edge. Only a small part of the plate was covered by the sector, but sufficient stars were included to give fairly satisfactory results, even though images within 1 cm. of the edge of the plate were not measured. Exposure times were 10 minutes and the sector gave not less than 10,000 flashes per minute.

Fourteen plates were taken in E3, the first series of eleven being of one area, and the final three of a second area having a considerable overlap with the first. The number of exposures on each plate varied from two to seven. They were made alternately with and without a sector. A further seven plates were taken in E6, but with two exposures only on each plate, in order to reduce to a minimum the possible effects of the sky fog, which has been ignored.

The sectors were measured geometrically and also checked by the inverse square law, with closely accordant results. The plates were measured and reduced both at Pretoria and at the Cape. At Pretoria the magnitudes of stars brighter than $8^{m}\cdot9$ were assumed from Fabry measures, and the reductions were made graphically. At the Cape the modified Schwarzschild method used in the earlier work for the reduction of the grating plates was applied, so that only a zero point and the reduction constants found from geometrical measurements of the sectors had to be assumed. The two sets of measures agreed well. The stars were not specially selected and, as so often happens in such a case, there is a correlation between magnitude and colour. Hence there is a danger that, in correcting the colour characteristics to those of the standard S magnitudes, the magnitude scale has been affected. Further, we have been obliged to assume, owing to insufficient data, that there is no variation of colour correction with magnitude.

Obtaining magnitudes $Pg(S')$ from the measured values by applying colour and zero corrections as follows:

$$\text{for E3, } Pg(\text{Sector}) - SPg = -0\cdot17\ (C.I. - 0\cdot64)$$
$$\text{,, E6, ,, } \quad -0\cdot13\ (C.I. - 1\cdot00),$$

the mean difference $Pg(S') - SPg$, taken without regard to sign, is $0^{m}\cdot056$. This indicates a mean error in $Pg(S')$ of about $0^{m}\cdot05$, but it should be noted that in fact the sector magnitudes were of varying reliability, some being means from several plates, others depending only on a single plate. Further, some of the exposures involved disturbing the plate holder. The general set-up was crude and conditions were not as satisfactory as in the regular programmes, so that the accuracy per exposure is lower. In so far as this exploratory investigation goes, there is no detectable difference of scale between the sector results and the standard SPg magnitudes (see Table VIII).

TABLE VIII

$Pg(Sector)' - SPg$

[Units $0^{m}\cdot01$, numbers of stars in parentheses]

m	7·25	7·5	8·0	8·5	9·0	9·5	10·0	10·25
E(3)	−6(1)	−3(4)	+5(5)	−1(8)	+2(6)	+3(10)	0(14)	−7(7)
E(6)	...	...	...	+8(2)	0(6)	0(6)	−2(9)	+3(7)
All	−6(1)	−3(4)	+5(5)	+1(10)	+1(12)	+2(16)	−1(23)	−2(14)

This method of obtaining a magnitude scale is applicable to almost any instrument for the range of magnitude which can be recorded with reasonably short exposures, provided that star fields can be found which are rich enough to provide a sufficient number of stars in the area covered by the sector. If it is desired to set up a magnitude sequence *ab initio* by the modified Schwarzschild method, the most convenient sector constant is about one magnitude; if it is desired to extend a known sequence, sector constants up to about five magnitudes can be conveniently employed, though a sector with such a large constant must be very carefully made. It must also be remembered that the light of the sky (and possibly other false light) constitutes a danger in all such methods of setting up a magnitude scale, as it is difficult to devise any simple and satisfactory method of correcting in-focus images for its effect.

A further trial of the rotating sector method is being made with the Cape Astrographic Refractor, in an attempt to extend the *SPg* sequence to the twelfth magnitude. A sector 15 cm. in diameter and having a reduction factor of approximately $2^{m}\cdot5$ is being used.

7.2. *Variable exposure times.*—Numerous investigators have used the method of variable exposure times for setting up magnitude scales. In particular, Halm used it extensively and has given a full account of it.* Seares also discusses it in his account of the photometric methods used with the Mount Wilson 60-inch reflector.†

Experiments made with Ilford Zenith plates showed that the ordinary characteristic curve, i.e. the photographic density measured over an extended area (*not* in a sharp star image) and plotted against magnitude, showed no detectable change in shape for exposures varying from three to ten minutes. In particular it was found that the curve for a ten-minute exposure coincided, within the accuracy of measurement, with that for a three-minute exposure when displaced about a magnitude along the abscissa axis. It was therefore natural to assume, despite the complexities involved in the formation of an in-focus photographic star image, that a photograph of a star field with the longer of these two exposures might give the same result as one with the shorter exposure but with all the stars apparently brighter by about one magnitude, and that a plate having two such exposures might be regarded as calibrated in much the same way as if it had received two exposures of equal length, the one with and the other without an absorbing screen or a rotating sector of known constant.

A trial of this method was made on two plates of E8. Each plate was cut into two portions. One portion was used for photographing E8 and received four exposures, the first and fourth being ten minutes, the second and third of three minutes ten seconds. The other portion received four calibrating exposures of the same lengths in a tube sensitometer, the light source of which was kept strictly constant. The two portions were then developed together. The measured difference between the calibration exposures was 1·01 magnitudes and this value was used for the reduction of the stellar observations by the modified Schwarzschild method. The colour difference between the magnitudes thus obtained, $Pg(R)$ and SPg, is not a simple linear one, but it can be fairly well represented by

$$\begin{aligned} Pg(R) - SPg &= 0\cdot20 - 0\cdot16\,C.I. \qquad (C.I. < 0\cdot81) \\ &= 0\cdot07 \qquad\qquad\qquad (C.I. > 0\cdot81). \end{aligned}$$

* *Magnitudes of stars contained in the Cape Zone Catalogue of* 20,843 *Stars*, London, 1927.

† F. E. Seares, *Mount Wilson Contributions*, **4**, No. 80, 1914.

Magnitudes corrected to the *SPg* colour system by this formula were denoted $Pg(R')$ and the mean residual $Pg(R')-SPg$, taken without regard to sign, was $0^{m}.042$. There is an apparent difference of scale (see Table IX) amounting to one or two per cent, but this is hardly significant with the numbers of stars observed.

TABLE IX

$Pg(R')-SPg$

[Units $0^{m}.01$]

m	7·25	7·5	8·0	8·5	9·0	9·5	10·0	10·5	10·25
$Pg(R')-SPg$	−2(2)	−3(5)	−1(9)	+1(6)	0(4)	+5(6)	+1(4)	+3(5)	+5(2)

The observational material on which this result is based is rather scanty, but it is sufficient to indicate the possibilities of the method, which appears to be both rapid and simple in use. It is, however, limited by the complications introduced by the light of the night sky and perhaps by the varied colours of the stars to be observed. The quantity measured by the Schilt photometer is not a simple measure of the photographic density but an average over the structure of the star image, which is itself a complicated function of the optical properties of the telescope and of the intricate physical properties of the photographic emulsion. The method can be used with full confidence only after an exhaustive examination of it has been made, and in particular a broader foundation is required than a mere demonstration of the constancy with respect to exposure time of the shape of the photographic characteristic curve, determined as in ordinary photographic photometry. It should also be noted that the fact that the method is entirely reliable with one photographic emulsion is no guarantee that it will be satisfactory with another.

8. *Concluding remarks.*—The rapid development of photoelectric photometry during the past few years may have led some astronomers to regard photographic photometry as being completely out-dated. It is true that, given a uniformly transparent sky, photoelectric methods can be much more accurate, and for some photometric problems are more convenient than orthodox photographic methods. There are, however, other branches of photometry for which the photographic method remains the most efficient. In fact, during recent years the importance of in-focus photometry has been increased rather than diminished by the development of wide-field lenses, and especially of the Schmidt telescope and related optical systems. For the moment astronomers are unable to utilize these new inventions to the full, for lack of adequate measuring machinery, but once this difficulty has been overcome, the photographic method must surely remain supreme for all problems depending on, or profiting by, the simultaneous observation of very large numbers of stars.*

As in other fields, we are likely to achieve the maximum efficiency when we regard the photoelectric and photographic methods as complementary. Thus

* We have mentioned the Schmidt telescope here, but do not wish to suggest that our results apply in quantitative detail to this or to other optical systems differing from the Cape cameras. For example, the colour behaviour of a Schmidt camera will differ in important respects from that of a refractor. A referee has suggested to us that the quality of Schmidt images may be so good that their rate of growth with magnitude will be too slow for accurate photometric measures, but we do not yet believe that this will in practice be the case with the focal lengths and apertures likely to be used for photometry. The problems we have discussed will crop up in most magnitude programmes based on in-focus star images, but for quantitative knowledge each astronomer must be prepared to spend much time investigating the behaviour of his own apparatus under his own working conditions.

the most efficient solution to one of the biggest and most important problems of photometry—that of finding on a single uniform system the magnitudes and colours of stars all over the sky—is probably to be found in a combination of different methods, the photoelectric or the Fabry method being used to set up standard sequences of stars in various areas, and then the photographic method to tie these sequences together and by interpolation to extend their magnitude system to the great body of the stars.

While some of the poor reputation that in-focus photometry has for accuracy can be attributed to the inherent difficulties of the subject, especially because these are concealed under a superficial appearance of simplicity, the greater part in our opinion must be attributed to the desire to get too many results out of too few observational data. It is regarded as essential to spend from twenty to thirty plates to get the parallax of a single star, yet some astronomers expect to get the magnitudes of several hundred stars from a single plate, and that plate not always obtained under the best conditions! Even after granting the meticulous care and precision that a modern trigonometrical parallax determination demands, we still maintain that such a discrepancy in observational effort is much too great.

To attain accuracy in in-focus photometry at least four conditions must be observed: (*a*) the plates used must match the optical properties of the telescope, (*b*) the images measured on any plate should be restricted to the optimum range of magnitude and to the central area for which the distance correction is either negligible or very well defined, (*c*) the colour of every star must be found, and every plate corrected individually for colour, (*d*) when two regions of the sky are compared, observations must be made under the best atmospheric conditions and at equal zenith distances.

(*a*) The need for condition (*a*) may be illustrated by some early plates taken at the Cape, whose range of spectral sensitivity was too long for the chromatic aberrations of the camera lenses, with the result that the reduction of the observations was complicated by a troublesome "photographic Purkinje effect". The advantages of using a wide spectrum range—increased speed for faint objects, less effect of absorption lines on the resulting magnitudes—must not be bought at the price of undue complications in the plate reductions.

(*b*) The range of magnitude measured on any one plate should be limited because there is an optimum exposure for stars of a given brightness, at which the measures are most accurate and at which the accidental effects of seeing, guiding, etc. reach a minimum.* The restriction to the central part of the field follows from the complicated behaviour of the distance correction, which seems to vary from plate to plate, and to depend not only on the distance from the centre but also on the density of the image and the colour of the star.† Possibly the field corrections for a Schmidt-type telescope would be less complicated than for a wide-field refractor. The corrections for the Cape Photometric Cameras have not yet been investigated carefully for distances from the centre greater than $2^{\circ}\cdot5$. Up to this distance they have been determined by comparison with measures by the Fabry method (which is free from such errors). They appear to be less than $0^{m}\cdot01$ up to 2° from the centre, and less than $0^{m}\cdot04$ at 3°, but beyond this they increase rapidly with distance. Thus, although the lenses were designed

* *The Observatory*, **67**, 108, 1947.

† e. g. see *Introduction to the Mount Wilson Polar Catalogue*, 1941.

for a 15° diameter field, only exceptional circumstances justify their use for determining magnitudes of stars more than 3° off axis. Whenever possible the measures should be confined within 2° of the centre of the field.

(*c*) As we have shown earlier, there is a considerable spread in colour equation even among plates of the same batch that have received similar treatment. In extreme cases the fluctuation exceeds 0·05 *C.I.*, so that a star having a colour index differing by $1^{m}\cdot0$ from the average would be assigned a magnitude more than $0^{m}\cdot05$ in error, unless the colour correction is evaluated separately for each individual plate. For this to be done satisfactorily, great care in the selection of standard stars is necessary: ideally they should be uniformly distributed over the required field, over the required magnitude range, and over the required colour range. It is astonishing how rarely an actual star field comes near satisfying these simple requirements. Measured magnitudes must always be reduced to some standard system if they are to be of general value, and thus in any case the colour of every individual star must be known.

(*d*) It should be unnecessary to stress the need to limit the comparisons of two regions of the sky to the times of best atmospheric transparency and to equal zenith distance. Atmospheric changes are the most important source of random errors in zero point determinations. If possible the zenith distance at which the comparison is made should not exceed 30°, and if it has to exceed this distance the regions being compared should if possible be less than 45° apart, on account of variation in atmospheric conditions with azimuth.

We have found that even on apparently good nights there is often a considerable patchiness in the sky, quite imperceptible to the naked eye, however experienced. The increased use of precise photoelectric apparatus is quickly confirming that the atmosphere imposes a limit on the accuracy with which any photometric observations can be made, a limit which can only be reduced by repeated observation. The extremely small standard errors which are frequently associated with photoelectric observations invariably refer to the results of rapid comparisons between objects that are close together in the sky, or to observations in different colours of the same object. It has yet to be proved that standard errors of $0^{m}\cdot01$ or better can be attained *as a routine* in a single observation by any method when the problem involves comparison of stars say 20° or more apart in the sky.

When the aim is to produce stellar magnitudes, rather than colours, or variable star light curves, etc., there is a natural limit to the accuracy which it is worth attempting to reach. This results from the vagueness inherent in the definition of a magnitude, which is not a quantity explicitly and precisely defined as a function of certain exact physical parameters. Translated into practical terms, the worth-while limit is the accuracy with which it is possible to transform magnitudes determined with one instrument under certain conditions to the system of magnitudes determined with another instrument, or with the same instrument under different conditions. The size of this limit is of fundamental importance, although it is not yet precisely known. Our experience so far with photographic and photovisual magnitudes suggests that, provided a parameter measuring colour is known for every star, this limit is of the order $0^{m}\cdot01$, even when there is a considerable difference in the methods by which the magnitudes have been measured. Whether the limit would remain as small as this if magnitudes such as *SPg*, which are based on measures which virtually exclude all light to the violet of 3800 A., were compared with magnitudes determined

with the aid of an aluminized reflector, using light down to 3000 A., is something which should be tested. It is possible that this would require an extra parameter measuring the absorption at the head of the Balmer series for every star. Similarly, we do not yet know the limit of accuracy with which a series of magnitudes determined photoelectrically can be compared with a similar series determined photographically. The answer presumably depends again on the range of spectrum used. Experience at Cambridge suggests that it may also be in the vicinity of $0^{m}\cdot01$.

It is a pleasure to express our thanks to Dr Jackson and Mr Cousins for their many helpful suggestions and for the very stimulating discussions we have had during the preparation of this paper.

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DYNAMIC EFFECTS OF A LIQUID CORE

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Summary

The effect of the inertia of the Earth's core on the bodily tides and related phenomena is examined. Previous discussions, which have neglected this inertia, are found to be sufficiently accurate for the semidiurnal, fortnightly, and semiannual tides, but to need modification for those depending on terms of the forms zx, zy in the gravitational potential. A theory of these is developed for an Earth model of Wiechert type, with a fluid core, and applied to the free variation of latitude and the lunar nutation. It was known that with a rigid shell a fluid core would produce a shortening of the free period, in contradiction with observation; but it is found that with a shell of reasonable rigidity ($1{\cdot}9 \times 10^{12}$ dynes/cm.^{2}) the motion induced in the core is much less than with a perfectly rigid shell, and that agreement between theory and observation can be obtained with a rigidity differing little from what was found satisfactory when the inertia of the core was neglected.

Fluidity of the core has been seen to be capable of producing a reduction of the amplitude of the lunar nutation about three times as large as is needed to account for the present discrepancy between theory and observation. Allowance for elasticity of the shell is found to reduce this; allowance for the considerable obliquity of the ecliptic further reduces the correction for the nutation in latitude, but increases that for the nutation in longitude. Even for the nutation in latitude the correction remains too large, but possibly not too large for the excess to be due to inaccuracies of the Wiechert model, and there is a need for a theory that takes account of the fuller knowledge now available on the distribution of density and rigidity in the Earth.

1. *Introduction.*—It has been seen * that the fluidity of the Earth's core is likely to have substantial effects on the amplitude of the lunar nutation and on the period of the free (Eulerian) nutation. The analysis was based on that of Poincaré † and Lamb ‡, which assumed a rigid shell, though Poincaré gave a qualitative discussion of the effect of elasticity of the shell at the end of his paper. The effect on the lunar nutation was three times the difference between the observed and calculated values. It appeared possible, however, that the effect would be smaller if account was taken of the elasticity of the shell.

Precession and nutation arise from diurnal components of the tidal potential. All components of the bodily tide have hitherto been treated by the method introduced by Kelvin and adapted by G. Herglotz to a heterogeneous Earth consisting of a shell and core, both taken elastic and uniform. Herglotz's theory was applied by me § to a liquid core, and the theory was extended to higher harmonics by L. Rosenhead.|| The procedure neglects inertial terms. This was justified for a solid Earth by the consideration that the slowest elastic vibrations have periods short compared with any of the astronomical periods, but its applicability to a

* H. Jeffreys, *M.N.*, **108**, 206–9, 1948.
† H. Poincaré, *Bull. Astr.*, **27**, 321–356, 1910.
‡ H. Lamb, *Hydrodynamics*, Cambridge, pp. 724–728, 1932.
§ H. Jeffreys, *Geophys. Suppl.*, **1**, 371–83, 1926.
|| L. Rosenhead, *Geophys. Suppl.*, **2**, 140–196, 1929.

fluid core is highly doubtful, since the rigidity then has to be taken zero, and inertial terms are no longer small compared to elastic ones. Consequently it is desirable to examine afresh the theory of all components of the bodily tide, retaining the inertial terms for motions in the core.

1.1. *Theory of the shell.*—In the first place, if the motions of the shell are prescribed at the outer surface and the core boundary, any displacement of the shell must be consistent with them and can therefore be regarded as a rigid body displacement with a superposed elastic displacement. Elastic stresses, being internal reactions, do not change the resultant angular momentum, so that the rigid body displacement can be chosen so as always to give the correct total angular momentum of the shell. Then the elastic stresses will correspond to wholly internal vibrations, and we can apply to them the principle that they must have nearly their statical values on account of the shortness of the free periods.

The possible displacements of a homogeneous shell on the statical theory can be classified into three types corresponding to a given solid harmonic K_n:

$$\phi \text{ type: } u_i = A\partial K_n/\partial x_i,$$

$$\omega \text{ type: } u_i = Br^2\partial K_n/\partial x_i + Cx_iK_n,$$

$$\chi \text{ type: } u_i = \epsilon_{ikm}x_k\partial K_n/\partial x_m.$$

The angular momentum associated with a displacement of the form $f(r)e^{i\gamma t}\partial K_n/\partial x_r$ is, apart from a factor $i\gamma e^{i\gamma t}$,

$$h_i = \epsilon_{ikm}\int\int\int \rho x_k f(r)\frac{\partial K_n}{\partial x_m}\,d\tau$$

$$= \epsilon_{ikm}\left[\int\int\int \rho x_k f(r) l_m K_n\,dS - \int\int\int \rho\delta_{km} f(r) K_n\,d\tau \right.$$

$$\left. - \int\int\int x_k \frac{x_m}{r}\frac{d}{dr}\{\rho f(r)\}K_n\,d\tau\right].$$

If the boundary is a sphere, $l_m = x_m/r$, and all terms are zero. The term Cx_iK_n, being wholly radial, also gives no resultant angular momentum. Hence the statical ϕ and ω types of solution, if applied to a shell with small ellipticities e, will give an angular momentum of order e times what would be expected if the displacements were in directions such that the components would be additive. But in the statical theory the product of e and the elastic displacements is already neglected in forming the boundary conditions. Hence we introduce no new type of error by supposing the ϕ and ω types to give no resultant angular momentum.

In the χ type each component displacement is a solid harmonic of degree n. When we form the angular momentum this is multiplied by a harmonic of degree 1 and integrated over a sphere; the integral will therefore vanish unless $n = 1$ or -2. But we have the expression for the stress components across a concentric sphere

$$rp_{ri} = px_i + \mu\left\{r\frac{\partial u_i}{\partial r} - u_i + \frac{\partial q}{\partial x_i}\right\}.$$

p and $q\,(=x_iu_i)$ vanish for the χ type, and hence all components p_{ri} can be zero only if u_i is homogeneous of degree 1; thus $n = 1$ and the displacement is a rigid body rotation. Hence if $n \neq 1$ the χ type is excluded by the condition that the

stress over the outer surface is zero, and if $n=1$ by the condition that the general rotation gives the correct angular momentum.

Hence the equations satisfied by the elastic displacements in the shell given by the existing statical theory are still satisfied if the body is free to rotate. The condition at the outer boundary is unaltered; it remains to be seen what change, if any, is needed at the inner boundary.

1.2. *Motion in the core.*—One new feature for the core is that we must retain inertia; but then the rotation terms in the equations indicate a special axis, which we may take by identifying the axis of z with that of the mean rotation. For ease of comparison with the shell we state the equations of motion as for displacements, but for these to be small we must take the axes of x and y to be rotating with the mean angular velocity ω. Another feature is that for some types of motion the disturbance of the transverse velocities far exceeds that of the normal velocity, and it may be incorrect to neglect the ellipticity of the core boundary. As p is used for the mean stress in the shell theory we use it for the fluid pressure with reversed sign. At present we retain the viscosity. Then the equations of motion, where u, v, w are now coordinates of a particle with respect to axes rotating with angular velocity ω about the axis of z, are

$$\left.\begin{aligned} \ddot{u} - 2\omega\dot{v} &= \frac{\partial U}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2\dot{u}, \\ \ddot{v} + 2\omega\dot{u} &= \frac{\partial U}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2\dot{v}, \\ \ddot{w} &= \frac{\partial U}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2\dot{w}. \end{aligned}\right\} \tag{1}$$

We assume a time factor $e^{i\gamma t}$ and put

$$x+iy=\zeta, \quad x-iy=\zeta^*, \quad u+iv=V, \quad u-iv=V^*, \quad U+\int\frac{dp}{\rho}=\Omega. \tag{2}$$

Then

$$\left.\begin{aligned} \ddot{V} + 2i\omega\dot{V} - \nu\nabla^2\dot{V} &= \frac{\partial\Omega}{\partial x} + i\frac{\partial\Omega}{\partial y} = 2\frac{\partial\Omega}{\partial\zeta^*}, \\ \ddot{V}^* - 2i\omega\dot{V}^* - \nu\nabla^2\dot{V}^* &= 2\frac{\partial\Omega}{\partial\zeta}, \\ \ddot{w} - \nu\nabla^2\dot{w} &= \frac{\partial\Omega}{\partial z}, \end{aligned}\right\} \tag{3}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\frac{\partial V}{\partial\zeta}+\frac{\partial V^*}{\partial\zeta^*}+\frac{\partial w}{\partial z}=0. \tag{4}$$

These lead to

$$(i\nu\nabla^2+\gamma)^2\nabla^2\Omega=4\omega^2\partial^2\Omega/\partial z^2. \tag{5}$$

The core viscosity appears to be small, and if we neglect it we have Poincaré's differential equation

$$\gamma^2\left(\frac{\partial^2\Omega}{\partial x^2}+\frac{\partial^2\Omega}{\partial y^2}\right)+(\gamma^2-4\omega^2)\frac{\partial^2\Omega}{\partial z^2}=0 \tag{6}$$

and

$$V=-\frac{2}{\gamma(\gamma+2\omega)}\frac{\partial\Omega}{\partial\zeta^*}, \quad V^*=-\frac{2}{\gamma(\gamma-2\omega)}\partial\Omega, \quad w=-\frac{1}{\gamma^2}\frac{\partial\Omega}{\partial z}. \tag{7}$$

Any polynomial of degree 1, 2 or 3 that satisfies (6) will also satisfy (5). With small non-zero viscosity, therefore, it should be possible to construct a boundary-layer theory to take account of dissipation. We are not directly concerned with this here, but it seems worth while to record the basic differential equation (5).

There are five linearly independent solutions of degree 2, which we may take to be

$$\Omega = \gamma^2 z\zeta; \qquad \gamma^2 z\zeta^*; \qquad \gamma^2\zeta^2; \qquad \gamma^2\zeta^{*2}; \qquad \gamma^2 z^2 - \tfrac{1}{2}(\gamma^2 - 4\omega^2)\zeta\zeta^*. \tag{8}$$

In this form they have the correct dimensions. With ω, γ positive, the first and third represent waves travelling in the negative sense with respect to axes fixed in the Earth, the second and fourth in the positive sense, while the last is independent of longitude. We need not consider the second and fourth specially, partly because they do not correspond to actual tidal components, and partly because if they are needed they can be derived from the first and third by reversing the sign of γ.

If the undisturbed boundary is

$$x^2 + y^2 + \frac{z^2}{(1-e)^2} = a^2, \tag{9}$$

the direction cosines of the normal are x/R, y/R, $z/(1-e)^2R$, where

$$R^2 = x^2 + y^2 + z^2/(1-e)^4 = a^2 + z^2\{1/(1-e)^4 - 1/(1-e)^2\}. \tag{10}$$

The normal component of the displacement must be continuous; to make it comparable with that of the shell we define

$$q = xu + yv + zw/(1-e)^2. \tag{11}$$

This will not affect the value taken for the shell, for in the shell we neglect the product of the elastic displacements and the ellipticity anyhow; the need to retain e for the core arises from the fact that w may greatly exceed the normal displacement.

We have then the following standard solutions, where real or imaginary parts may be taken when required. A factor $e^{i\gamma t}$ is understood throughout.

(*a*) $\Omega = \gamma^2 z\zeta = \gamma^2 z(x+iy)$,

$$u = -\frac{\gamma}{\gamma - 2\omega}z, \quad v = -\frac{i\gamma}{\gamma - 2\omega}z, \quad w = -(x+iy),$$

$$q = -\left(\frac{\gamma}{\gamma - 2\omega} + \frac{1}{(1-e)^2}\right)z(x+iy).$$

(*b*) $\Omega = \tfrac{1}{2}\gamma^2\zeta^2 = \tfrac{1}{2}\gamma^2(x+iy)^2$,

$$u = -\frac{\gamma}{\gamma - 2\omega}(x+iy), \quad v = -\frac{i\gamma}{\gamma - 2\omega}(x+iy), \quad w = 0,$$

$$q = -\frac{\gamma}{\gamma - 2\omega}(x+iy)^2.$$

(*c*) $\Omega = \gamma^2 z^2 - \tfrac{1}{2}(\gamma^2 - 4\omega^2)(x^2+y^2)$,

$$u = x - \frac{2i\omega}{\gamma}y, \quad v = y + \frac{2i\omega}{\gamma}x, \quad w = -2z,$$

$$q = x^2 + y^2 - 2z^2/(1-e)^2.$$

In the absence of rotation, and with $e=0$, all these solutions could be treated together, all in fact giving $\Omega = -\frac{1}{2}\gamma^2 q$. But in actual conditions the ratios are very different. Terms of the form (*a*) in the potential correspond to the diurnal terms, with γ near ω, and the free nutation, with γ/ω small. The semidiurnal group is represented by (*b*) and the fortnightly and other long-period tides by (*c*).

Now taking $\Omega=0$ is equivalent to neglecting both rigidity and inertia in the core. In the stress equation at the core boundary q enters in combination with factors of the order of g/a. Hence if $\Omega \simeq \omega^2 q$ or less,

$$\frac{\Omega}{gq/a} \simeq \frac{a\omega^2}{g} \simeq e$$

and the neglect of Ω in the stress equation produces errors of the order of those already made in applying the theory for a spherical shell to the actual shell. Then it follows that this neglect is justified to the accuracy attempted for the free nutation ((*a*) with γ/ω small) and the semidiurnal tides ((*b*) with γ near 2ω). For case (*c*) γ/ω is small. Ω departs considerably from a solid harmonic, and in comparison with the solution for the shell it must be replaced by the sum of a zonal harmonic and a term in r^2. A similar separation has to be made in the theory of this term in all tidal problems, the symmetrical part being ignored because it produces no observable effect—no effect at all in the case of incompressibility. The relevant part of Ω is the variation of Ω from pole to equator, which is about $\frac{2}{3}\omega^2$ times that of q, and the neglect of Ω is again justified. In cases (*b*) and (*c*), again, the core contributes nothing to the total angular momentum. Hence, *the existing theory is valid for the semidiurnal, fortnightly and semiannual tides.*

For the free nutation neglect of Ω is still valid. For the diurnal tides it is not, for if γ is near ω the coefficient in q is of order e, so that Ω may be of the same order as gq/a. The conditions recall those in the theory of the diurnal tide in a uniform ocean, where it is known that the whole disturbance is in the currents and not in the elevation.

The conditions at the core boundary in these problems need some restatement. As usual, if we define $q(a)/a$ as the radial elastic displacement at the outer surface and $q(a_1)/a_1$ as that at the core boundary, then existing theory gives the whole of the elastic displacements in the shell in terms of $q(a)$ and $q(a_1)$. For the shell it is adequate to identify radial and normal displacements. But the q that we have found for the core is r times the whole displacement, and therefore is not equal to $q(a_1)$ of the shell theory, but to $q(a_1)$ together with a_1 times the normal displacement arising from the rotation of the shell. This applies whenever there is a shift of the axis of rotation. It is to this that the reduction of the theoretical amplitude of the nutation is due, since the difficulty of producing large normal motions in the core when γ is near $-\omega$ will be associated with a set of couples opposing changes of rotation of the shell. This makes the analysis much more difficult for motions of the type (*a*), and a different procedure takes the various complications into account more satisfactorily.

2.0. *Theory of changes of rotation.*—We are here concerned entirely with cases where changes of motion in the core are of type (*a*), and the external potential, if any, consists of terms of the forms zx, zy. It is convenient to make use of a principle given by Poincaré in the paper quoted, but not reproduced by Lamb. To any order of accuracy the velocities in the fluid under a second harmonic

disturbance are linear functions of the coordinates, and therefore the coordinates of a particle of the fluid at any instant are linear functions of those in a standard position, which we may take to be a position of symmetry about the z axis. Then if suffix zero refers to the standard position we shall have at any time

$$x_i = x_{i0} + \alpha_{ik} x_{k0}, \tag{1}$$

where α_{ik} in our problem will be permanently small functions of the time. The boundary condition is that the normal velocity is continuous with that of the shell. The boundary remains an ellipsoid. Displacements arising from the harmonics x^2-y^2, xy, $x^2+y^2-2z^2$ alter the lengths of the principal axes and are wholly determined by the corresponding changes of the shell. As they are independent of those due to terms in xz, yz, we need not consider them further. (By including them we could derive the results of 1.2 in another way.) For displacements that rotate the principal axes but do not alter their lengths a, a, c, we can build up the general displacement in the following way. First multiply all z for the core and shell by a/c. Then apply an arbitrary rotation to the core without moving the shell; this is possible because the boundary is now spherical. Next multiply all z by c/a. This brings the boundary back to its original position. Finally rotate the shell and core together. All steps satisfy the boundary conditions; the axes are unaltered in length; and displacements of the core and shell are given by six independent parameters, of which three specify the rotation of the shell. Now in (1) the nine quantities α_{ik} are connected by three relations specifying that the axes are unaltered in length, so that not more than six of them can vary independently. It is therefore possible to choose six of the α_{ik} and use them as Lagrangian coordinates. This procedure has the convenient feature that integrations are through the undisturbed position. The kinetic energy, however, contains a term in $\omega^2(x^2+y^2)$, and to find it to the second order in the displacements we need some of the α_{ik} to the second order.

If the coordinates of a particle in the standard position are x_0, y_0, z_0, those after the first expansion are

$$x_1, y_1, z_1 = x_0, y_0, \frac{a}{c} z_0. \tag{2}$$

The transformation for a general rotation can be expressed, to the second order, by

$$\left.\begin{aligned} x_2 &= (1-\tfrac{1}{2}l^2-\tfrac{1}{2}n^2)x_1 - (n+\tfrac{1}{2}lm)y_1 + lz_1, \\ y_2 &= (n-\tfrac{1}{2}lm)x_1 + (1-\tfrac{1}{2}m^2-\tfrac{1}{2}n^2)y_1 + mz_1, \\ z_2 &= (-l+mn)x_1 - (m+ln)y_1 + (1-\tfrac{1}{2}l^2-\tfrac{1}{2}m^2)z_1. \end{aligned}\right\} \tag{3}$$

The coordinates after the contraction are

$$x_3, y_3, z_3 = x_2, y_2, \frac{c}{a} z_2. \tag{4}$$

The final position is then found by putting

$$\mathbf{x} = \mathbf{x}_4 = \boldsymbol{l}' \mathbf{x}_3, \tag{5}$$

where $\boldsymbol{l}'$ is the result of substituting l', m', n' for l, m, n in (3). The result is (1) with

$$\left.\begin{aligned}
\alpha_{11} &= -\tfrac{1}{2}l^2 - \tfrac{1}{2}l'^2 - \tfrac{1}{2}(n+n')^2 - \frac{c}{a}ll', \\
\alpha_{12} &= -n - n' - \tfrac{1}{2}lm - \tfrac{1}{2}l'm' - \frac{c}{a}l'm, \\
\alpha_{13} &= \frac{a}{c}l + l' - \frac{a}{c}mn', \\
\alpha_{21} &= n + n' - \tfrac{1}{2}lm - \tfrac{1}{2}l'm' - \frac{c}{a}lm', \\
\alpha_{22} &= -\tfrac{1}{2}m^2 - \tfrac{1}{2}m'^2 - \tfrac{1}{2}(n+n')^2 - \frac{c}{a}mm', \\
\alpha_{23} &= \frac{a}{c}m + m' + \frac{a}{c}ln', \\
\alpha_{31} &= -\frac{c}{a}l - l' + \frac{c}{a}mn - m'n + m'n', \\
\alpha_{32} &= -\frac{c}{a}m - m' - \frac{c}{a}ln + l'n - l'n', \\
\alpha_{33} &= -\tfrac{1}{2}l^2 - \tfrac{1}{2}m^2 - \tfrac{1}{2}l'^2 - \tfrac{1}{2}m'^2 - \frac{a}{c}(ll' + mm').
\end{aligned}\right\} \tag{6}$$

The kinetic energy of the core is to the second order (where μ for a moment denotes the mass of a particle)

$$\begin{aligned}
T &= \tfrac{1}{2}\Sigma\mu\{(\dot{x} - \omega y)^2 + (\dot{y} - \omega x)^2 + \dot{z}^2\} \\
&= \tfrac{1}{2}A_{11}\{\dot{\alpha}_{21}^2 + \dot{\alpha}_{12}^2 + \dot{\alpha}_{31}^2 + \dot{\alpha}_{32}^2 + 2\omega(\dot{\alpha}_{21} - \dot{\alpha}_{12}) \\
&\quad + \omega^2(2 + 2\alpha_{11} + 2\alpha_{22} + \alpha_{12}^2 + \alpha_{21}^2)\} + \tfrac{1}{2}A_{33}\{\dot{\alpha}_{13}^2 + \dot{\alpha}_{23}^2 \\
&\quad - 2\omega(\dot{\alpha}_{13}\alpha_{23} - \dot{\alpha}_{23}\alpha_{13}) + \omega^2(\alpha_{23}^2 + \alpha_{13}^2)\},
\end{aligned} \tag{7}$$

where

$$A_{11} = \Sigma\mu x_0^2, \quad A_{33} = \Sigma\mu z_0^2. \tag{8}$$

In terms of the moments of inertia A_1, A_1, C_1 of the core

$$A_{11} = \tfrac{1}{2}C_1, \quad A_{33} = \tfrac{1}{2}(2A_1 - C_1). \tag{9}$$

We also write

$$F = \frac{c}{a}C_1 = 2\frac{c}{a}A_{11}; \quad F^2 = C_1(2A_1 - C_1). \tag{10}$$

Substituting in (7) and dropping terms above the second order and total derivatives with regard to the time we find

$$\begin{aligned}
2T &= C_1\omega^2 - (C_1 - A_1)\omega^2(l'^2 + m'^2) + C_1\omega(l\dot{m} - \dot{l}m) \\
&\quad + F\omega(l\dot{m}' + l'\dot{m} - m\dot{l}' - m'\dot{l}) + \omega(2A_1 - C_1)(l'\dot{m}' - \dot{l}'m') \\
&\quad + C_1(\dot{n} + \dot{n}')^2 + A_1(\dot{l}'^2 + \dot{m}'^2 + \dot{l}^2 + \dot{m}^2) + 2F(\dot{l}\dot{l}' + \dot{m}\dot{m}').
\end{aligned} \tag{11}$$

Three checks are available. The potential energy and the kinetic energy of the shell are independent of l, m, n, and we can form Lagrange's equations for l, m, n

at once. These give

$$\dot{n}+\dot{n}' = \text{constant}, \tag{12}$$

$$A_1\ddot{l}+F\ddot{l}'-C_1\omega\dot{m}-F\omega\dot{m}'=0, \tag{13}$$

$$A_1\ddot{m}+F\ddot{m}'+C_1\omega\dot{l}+F\omega\dot{l}'=0. \tag{14}$$

The first is a known result. But we must also have

$$\ddot{x}-2\omega\dot{y}-\omega^2x=\frac{\partial\Omega}{\partial x}\,;\quad \ddot{z}=\frac{\partial\Omega}{\partial z}. \tag{15}$$

By cross-differentiating and substituting from (6) we derive a multiple of (13). Hence (13) and (14) are equivalent to Helmholtz's equations.

The third check is got by taking $l=m=0$. Then T becomes applicable to a symmetrical rigid body under no forces. In this case, by forming and solving the equations for l', m', we derive at once the periods for small oscillations (referred, of course, to axes rotating with angular velocity ω)

$$l'+im' \propto e^{i\gamma t};\quad \gamma=0,\ 0,\ -\omega \text{ or } (C_1-A_1)\omega/A_1.$$

2.1. *Kinetic energy of a rigid shell.*—This can be derived by putting $l=m=0$ in (11). Taking the moments of inertia of the shell to be (A_0, A_0, C_0) we have, therefore,

$$\begin{aligned}2T_{\text{shell}} = {} & C_0\omega^2-(C_0-A_0)\omega^2(l'^2+m'^2)+\omega(2A_0-C_0)(l'\dot{m}'-m'\dot{l}')\\ & +C_0\dot{n}'^2+A_0(\dot{l}'^2+\dot{m}'^2).\end{aligned} \tag{16}$$

Then for the whole body, if we put $A=A_0+A_1$, $C=C_0+C_1$, and drop a constant,

$$\begin{aligned}2T = {} & A(\dot{l}'^2+\dot{m}'^2)+(C-C_1)\dot{n}'^2+C_1(\dot{n}+\dot{n}'^2)+\omega(2A-C)(l'\dot{m}'-\dot{l}'m')\\ & -(C-A)\omega^2(l'^2+m'^2)+A_1(\dot{l}^2+\dot{m}^2)+2F(\dot{l}\dot{l}'+\dot{m}\dot{m}')\\ & +C_1\omega(l\dot{m}-\dot{l}m)+F\omega(l\dot{m}'+l'\dot{m}-m\dot{l}'-m'\dot{l}).\end{aligned} \tag{17}$$

We have not yet formed the potential energy, but this form of T as it stands is applicable to the motion of a rigid shell and fluid core under no forces. If we put

$$l+im=\zeta,\quad l'+im'=\zeta', \tag{18}$$

Lagrange's equations are found to be equivalent to

$$\dot{n}=\text{constant},\quad \dot{n}'=\text{constant}, \tag{19}$$

$$(\gamma+\omega)[\{A\gamma-(C-A)\omega\}\zeta'+F\gamma\zeta]=0, \tag{20}$$

$$\gamma[F(\gamma+\omega)\zeta'+(A_1\gamma+C_1\omega)\zeta]=0. \tag{21}$$

The period equation is

$$\{A\gamma^2+(2A-C)\omega\gamma-(C-A)\omega^2\}(A_1\gamma^2+C_1\omega\gamma)=C_1(2A_1-C_1)(\gamma^2+\omega\gamma)^2. \tag{22}$$

Two roots are $\gamma=0$, $\gamma=-\omega$; the other two satisfy

$$\{A\gamma-(C-A)\omega\}(A_1\gamma+C_1\omega)=C_1(2A_1-C_1)\gamma(\gamma+\omega). \tag{23}$$

This is Lamb's period equation (28) and provides a further check on the algebra. But as Lamb works in terms of the angular velocities and not the displacements his equation does not give the roots $\gamma=0, \omega$. $\gamma=0$ is found on substitution to give $\zeta'=0$, while ζ may not be 0. The shell is symmetrical about the z axis, while the core has an arbitrary initial displacement and maintains it. The root $\gamma=-\omega$

gives $\zeta=0$ and represents a state with the whole body rotating rigidly about an axis different from that of z. These two solutions are therefore essentially trivial for free vibrations, but their existence is important in forced vibrations. The other roots, if $C-A$ and C_1-A_1 are small, are near 0 and $-\omega$. As might be expected, γ small gives $\zeta/\zeta' \doteqdot -1$, so that the core has little motion with reference to inertial axes. The change of the free period arises from the fact that F is an appreciable fraction of A. On the other hand, if γ is near $-\omega$ we get $\zeta/\zeta' \doteqdot -C/F$, which is large. The displacements of the shell in this motion are nearly compensated by much larger opposite changes in the core.

The existence of roots nearly equal in pairs is connected with what Poincaré calls double resonance. If there is an applied couple $Le^{i\gamma t}$ with $\gamma=-\omega+n$ and n small, ζ' will be zero when $A_1\gamma+C_1\omega=0$, that is,

$$\frac{n}{\omega}=-\frac{C_1-A_1}{A_1}\doteqdot-\frac{1}{400}. \tag{24}$$

On the other hand, there will be resonance if

$$(C\omega-A_n)\{(C_1-A_1)\omega+A_1n\}=C_1(2A_1-C_1)n(\omega-n), \tag{25}$$

the small root of which is nearly given by

$$\left(1-\frac{C_1}{C}\right)\frac{n}{\omega}=-\frac{C_1-A_1}{A_1}\,;\quad \frac{n}{\omega}=-\frac{9}{8}\cdot\frac{1}{400}. \tag{26}$$

The effect of the fluid core is therefore extraordinary for applied couples with retrograde motion in space and periods of order 1 year. With a slight change of period the amplitude of the motion of the shell can range from 0 to ∞. We have

$$\zeta'=-Le^{i\gamma t}\Big/\left\{C\omega-An-\frac{C_1(2A_1-C_1)n(\omega-n)}{(C_1-A_1)\omega+A_1n}\right\}. \tag{27}$$

If $|n/\omega|$ is much greater than $\frac{1}{400}$ and much less than 1,

$$\zeta'\doteqdot-Le^{i\gamma t}/\{C\omega-An-(C_1/A_1)(2A_1-C_1)\omega\}, \tag{28}$$

so that the effect of the fluid core is to increase the amplitude roughly by a factor of $\frac{1}{8}$. This should be relevant to the solar semi-annual and lunar fortnightly nutations. The reduction of amplitude found for the lunar nutation applies to cases where n/ω is negative and less numerically than $(C_1-A_1)/A_1$.

2.2. *Kinetic energy of an elastic shell.*—With a slight change of notation let x_i indicate the standard position of a particle, ξ_i the actual position. Let the angular velocity components corresponding to a rigid rotation be ω_i, and let the additional elastic displacement be u_i. Then the velocity v of a particle is given by

$$\begin{aligned} v^2&=(\dot\xi_i+\epsilon_{ikm}\omega_k\xi_m)(\dot\xi_i+\epsilon_{ips}\omega_p\xi_s)\\ &=\dot\xi_i^2+2\epsilon_{ikm}\omega_k\xi_m\dot\xi_i+\omega_k^2\xi_m^2-(\omega_k\xi_k)^2. \end{aligned} \tag{29}$$

Then $\dot\xi_i=\dot u_i$. Terms in products of the u_i can in general be neglected. For their ratios to corresponding parts of the elastic strain energy will be of order $(\gamma a/\beta)^2$ or $(\omega a/\beta)^2$, whichever is the larger, where β is the velocity of transverse elastic waves. But a/β is of the order of 40 minutes, and for periods of a day or more this ratio will be of order 10^{-3}. The second term will contribute a term to $2T$, linear in the $\dot u_i$, equal to

$$2\epsilon_{ikm}\int\int\int\rho\omega_k x_m\dot u_i\,d\tau. \tag{30}$$

In the elastic displacement u_i has the form

$$u_i = F(r)\frac{\partial K_n}{\partial x_i} + G(r)x_i K_n + H(r)\epsilon_{ips}x_p\frac{\partial K_n}{\partial x_s}. \tag{31}$$

The second term contributes 0 to (30). For the first, we notice that each component of $\epsilon_{ikm}x_m\partial K_n/\partial x_i$ is a spherical harmonic of degree n (here $n=2$) and gives 0 when integrated over a sphere. Hence this term, integrated through the shell, will contain the ellipticity as a factor, and can be neglected on the ground that we neglect products of the ellipticities and the elastic displacements. Again, if $n\neq 1$, the third term would give a tangential stress over a concentric sphere. Since there is no tangential stress over the outer boundary, the third term must contain the ellipticity as a factor. If $n=1$ with $H(r)$ constant, the third term is already of the form of the rotational terms in the velocity components. $H(r)$ might contain a term in r^2 owing to stresses arising from the variation of the rotation with time, but this again varies only on account of the ellipticities. Hence the constant part of $H(r)$, with $n=1$, can be absorbed into the general rotation, and all the other linear parts arising from this term are negligible.

If the u_i are zero the third and fourth terms of (29) will give on integration the kinetic energy of a rigid body, which we have already found. ω_1 and ω_2 are of the first order and their products by u_i will be of the second order. The customary method is to notice that these terms depend on position and not on changes of coordinates, and it is therefore indifferent whether we regard them as belonging to the kinetic energy or to the work function. Then they may be regarded as a contribution to the gravitational potential.

Now

$$\begin{aligned}(\omega_1^2+\omega_2^2+\omega_3^2)(\xi_1^2+\xi_2^2+\xi_3^2)-(\omega_1\xi_1+\omega_2\xi_2+\omega_3\xi_3)^2\\ =\omega_3^2(\xi_1^2+\xi_2^2)-2\omega_3\xi_3(\omega_1\xi_1+\omega_2\xi_2)+O(\omega_1^2,\omega_2^2).\end{aligned} \tag{32}$$

Forces arising from the first part are independent of the time, and in fact are cancelled by other terms arising from the disturbance of the gravitational field by the permanent ellipticity. Terms in the last batch that contain u_i are of the third order. Hence, in Cartesian notation, we can allow for the whole contribution of the elasticity of the shell to the kinetic energy by adding a term $-\omega z(\omega_1 x+\omega_2 y)$ to the gravitational potential.

In terms of our standard specification

$$\omega_1 = -\dot{m}' - l'\omega, \quad \omega_2 = \dot{l}' - m'\omega$$

and the equivalent extra gravitational potential is

$$\omega z\{(\dot{m}'+l'\omega)x-(\dot{l}'-m'\omega)y\}. \tag{33}$$

If $l'+im'\propto e^{-i\omega t}$ the coefficients vanish. Hence this term is zero for the precession. But for the Eulerian nutation it is important because this term is the primary reason why there are any elastic strains at all (there being no external forces).

2.3. *Correction of the boundary condition.*—The deformation of the core boundary affects the boundary condition for the core. For a rigid shell we find the normal velocity

$$V_n = -R\left(\frac{1}{c_1^2}-\frac{1}{a_1^2}\right)z(\dot{l}'x+\dot{m}'y). \tag{34}$$

If the normal strain displacement is

$$\frac{Rz}{a_1^2}(s_1x+s_2y), \tag{35}$$

the time-derivative of this must be added to (34) to give the whole normal velocity of the boundary. The normal velocity of the core must be continuous with this sum. We can adjust this by replacing l', m' in the expression for the kinetic energy of the core by l'', m'', where

$$\begin{aligned}(l'',m'') &= (l',m') - \frac{c_1^2}{a_1^2-c_1^2}(s_1,s_2)\\ &= (l',m') - \frac{A_1-\frac{1}{2}C_1}{C_1-A_1}(s_1,s_2).\end{aligned} \tag{36}$$

Evidently with small ellipticity of the core a small deformation of the shell may make a considerable difference to the motion of the core.

2.4. *The work function.*—This includes (1) the work done by the external field, which includes (1 *a*) a part due to the rotation and (1 *b*) a part due to the elastic strain, but this must be reduced by (2) the change of potential energy due to the mutual gravitation in the Earth, and (3) the energy of the elastic strain.

Let the gravitational potential of the external field be

$$U_0 = z(c_1x+c_2y). \tag{1}$$

Then the work function due to rotation is

$$\begin{aligned}W_{1a} &= \int\int\int \rho z(c_1x+c_2y)\,d\tau\\ &= \int\int\int \rho(z_0-l'x_0-m'y_0)\{c_1(x_0-n'y_0+l'z_0)\\ &\qquad +c_2(y_0+n'x_0+m'z_0)\}\,d\tau.\end{aligned} \tag{2}$$

The l, m, n displacement in the core makes no contribution because it does not alter the region occupied by the core material. The terms of the second order are (c_1, c_2 being regarded as of the first order)

$$W_{1a} = -(C-A)(c_1l'+c_2m'). \tag{3}$$

The contributions (1 *b*), (2), (3) can be derived from the Herglotz-Rosenhead theory, since the stresses calculated in that theory should lead directly to the change of potential energy for given deformations of the boundaries. Unfortunately, in the actual presentation certain constant factors have been suppressed, and cannot be restored without reworking a considerable part of the theory. In fact the first of the final pair of equations of the theory refers to continuity of stress over the inner boundary, the second to the outer boundary, but even this is not explicitly stated.

If we take the external potential in the general form

$$U_0 = k_nK_n = k_nr^nS_n \tag{4}$$

and the radial displacement to be $q(r)K_n/r$, we can take

$$q(r) = (Ar^2+B)(a_1/r)^{2n+1}+Cr^2+D. \tag{5}$$

As usual we take the outer radius to be a, core radius $a_1 = a\alpha$, rigidity of shell $= \mu$, $q(a) = q$, $q(a_1) = q_1$, density of shell $= \rho$, density of core $\rho(1+\eta)$. The material is supposed incompressible. Then the conditions on q at the surfaces and the absence of shear stress over them determine A, B, C, D in terms of q, q_1. We write the elastic displacement as

$$u_i = F(r)\frac{\partial K_n}{\partial x_i} + G(r)x_i K_n. \tag{6}$$

The value of $F(a)$ is important in matters connected with tidal deflections of the vertical, and is found to be, for $n=2$,

$$F(a) = \left\{\frac{3}{10} + \frac{\alpha^3(8-11\alpha^4+3\alpha^7)}{10(1-\alpha^3)(1-\alpha^7)}\right\} q - \frac{\alpha^3(8-11\alpha^2+11\alpha^5-8\alpha^7)}{10(1-\alpha^3)(1-\alpha^7)}\, q_1. \tag{7}$$

The part of the normal stress p_{rr} that depends on the rigidity is $\tau K_n/r^2$, with

$$\tau(a) = L_1 q - L_2 q_1, \quad \tau(a\alpha) = L_3 q - L_4 q_1, \tag{8}$$

and for $n=2$

$$\begin{aligned} L_1 &= \tfrac{1}{5}\mu\left(19 + \frac{\alpha^3(64-21\alpha^4-43\alpha^7)}{(1-\alpha^3)(1-\alpha^7)}\right); \\ L_2 &= \tfrac{1}{5}\mu\alpha^3\,\frac{64(1-\alpha^7)-21(\alpha^2-\alpha^5)}{(1-\alpha^3)(1-\alpha^7)}; \quad L_3 = L_2/\alpha^5; \\ L_4 &= \frac{\mu}{5\alpha^2}\left(19 - \frac{64(1-\alpha^7)-21(1-\alpha^3)}{(1-\alpha^3)(1-\alpha^7)}\right). \end{aligned} \tag{9}$$

The gain of elastic energy in small changes δq, δq_1 is

$$\delta V_e = \int\int (L_1 q - L_2 q_1)\frac{\delta q}{a^3} K_n^2\, dS - \int\int (L_3 q - L_4 q_1)\frac{\delta q_1}{a_1 a^2} K_n^2\, dS_1. \tag{10}$$

Put

$$\int\int S_n^2\, d\omega = M. \tag{11}$$

Then

$$\delta V_e = Ma^{2n-1}(L_1 q\delta q - L_2 q_1\delta q - \alpha^{2n+1}L_3 q\delta q_1 + \alpha^{2n+1}L_4 q_1\delta q_1),$$

which is an exact differential, since $L_2 = \alpha^{2n+1}L_3$. Hence

$$V_e = \tfrac{1}{2}Ma^{2n-1}(L_1 q^2 - 2\alpha^{2n+1}L_3 qq_1 + \alpha^{2n+1}L_4 q_1^2). \tag{12}$$

The energy of the displacement of the outer surface against the general gravitational field is

$$\tfrac{1}{2}g\rho\int\int \frac{q^2}{a^2} K_n^2\, dS = \tfrac{1}{2}Ma^{2n}g\rho q^2, \tag{13}$$

and that for the interface is

$$\tfrac{1}{2}Ma^{2n}\alpha^{2n+1}\frac{g(1+\eta)}{1+\eta\alpha^3}\rho\eta q_1^2. \tag{14}$$

On account of the mutual attraction between the disturbances this must be reduced by

$$\frac{1}{2}\frac{3gM\rho}{2n+1}\frac{a^{2n}}{1+\eta\alpha^3}(q^2 + 2\eta\alpha^{2n+1}qq_1 + \eta^2\alpha^{2n+1}q_1^2). \tag{15}$$

The work on the strains done by the external field is

$$\int\int k_n r^n S_n \cdot \frac{\rho q}{a} r^n S_n \cdot dS + \int\int k_n r^n S_n \cdot \frac{\rho\eta q_1}{\alpha a} r^n S_n \cdot dS_1 = M k_n \rho a^{2n+1}(q + \eta\alpha^{2n+1} q_1). \tag{16}$$

Collecting the contributions we find that the potential energy associated with the elastic strains is

$$V = \tfrac{1}{2} M\rho a^{2n} g(H_3 q^2 - 2\eta\alpha^{2n+1} H_1 q q_1 + \eta\alpha^{2n+1} H_2 q_1^2 - \frac{2ak_n}{g} q - 2\alpha^{2n+1}\eta \frac{ak_n}{g} q_1), \tag{17}$$

where H_1, H_2, H_3 are the coefficients in Rosenhead's equations, after putting his $\delta = 0$, $\epsilon = \infty$, except that, in his H_2, α^{2n+1} should be α^{2n-1}. We have given L_1, etc. in full only for $n = 2$*; but with this correction Rosenhead's analysis will give V for a general n.

Incidentally this analysis shows that the mysterious-looking relation $H_4 = \eta\alpha^{2n+1} H_1$ of Rosenhead's result (equation (99)) (generalized from Herglotz) is simply a consequence of the existence of a potential energy.

In detail, if $n = 2$, and

$$\sigma = \mu/g\rho a, \tag{18}$$

$$\left.\begin{aligned} H_3 &= 1 - \frac{3}{5(1+\eta\alpha^3)} + \tfrac{1}{5}\sigma\left[19 + \frac{\alpha^3(64 - 21\alpha^4 - 43\alpha^7)}{(1-\alpha^3)(1-\alpha^7)}\right], \\ \eta H_1 &= \frac{3\eta}{5(1+\eta\alpha^3)} + \frac{\sigma}{5\alpha^2}\frac{64(1-\alpha^7) - 21(\alpha^2 - \alpha^5)}{(1-\alpha^3)(1-\alpha^7)}, \\ \eta H_2 &= \frac{\eta(1 + \frac{2}{5}\eta)}{1+\eta\alpha^3} + \frac{\sigma}{5\alpha^2}\left(-19 + \frac{43 + 21\alpha^3 - 64\alpha^7}{(1-\alpha^3)(1-\alpha^7)}\right). \end{aligned}\right\} \tag{19}$$

If $K_2 = zx$ or zy, $M = \frac{4}{15}\pi$.

2.5. *Lagrangian equations of motion.*—We now take for the external potential $U_0 = z(c_1 x + c_2 y)$ as in 2.4 (1), and take for the outer and inner boundaries

$$qK_2 = r_1 zx + r_2 zy, \qquad q_1 K_2 = s_1 zx + s_2 zy. \tag{1}$$

Then we find for the Lagrangian function

$$\begin{aligned} 2L = &-(C_1 - A_1)\omega^2(l''^2 + m''^2) + C_1\omega(l\dot{m} - \dot{l}m) + F\omega(l\dot{m}'' + l''\dot{m} - m\dot{l}'' - m''\dot{l}) \\ &+ \omega(2A_1 - C_1)(l''\dot{m}'' - \dot{l}''m'') + A_1(\dot{l}''^2 + \dot{m}''^2 + \dot{l}^2 + \dot{m}^2) + 2F(\dot{l}\dot{l}'' + \dot{m}\dot{m}'') \\ &- (C_0 - A_0)\omega^2(l'^2 + m'^2) + \omega(2A_0 - C_0)(l'\dot{m}' - m'\dot{l}') + A_0(\dot{l}'^2 + \dot{m}'^2) \\ &- 2(C - A)(c_1 l' + c_2 m') \\ &- \frac{C}{2(1+\eta\alpha^5)}\left[\frac{g}{a}\{H_3(r_1^2 + r_2^2) - 2\eta\alpha^5 H_1(r_1 s_1 + r_2 s_2) + \eta\alpha^5 H_2(s_1^2 + s_2^2)\}\right. \\ &\left. - 2\{c_1 + \omega(\dot{m}' + l'\omega)\}(r_1 + \eta\alpha^5 s_1) - 2\{c_2 - \omega(\dot{l}' - m'\omega)\}(r_2 + \eta\alpha^5 s_2)\right], \end{aligned} \tag{2}$$

* The misprint, replacing $21\alpha^3$ by $21\alpha^5$, is in Herglotz's paper. It has survived both my reworking in 1913–14, which was deliberately undertaken on account of a suspicion that there might be a misprint in the final equations, and Rosenhead's recalculation for a general n.

with the restrictions

$$(l'', m'') = (l', m') - \frac{A_1 - \frac{1}{2}C_1}{C_1 - A_1}(s_1, s_2). \tag{3}$$

We introduce a pair of undetermined multipliers λ, μ, and put

$$l + im = \zeta; \quad l' + im' = \zeta'; \quad l'' + im'' = \zeta''; \quad \lambda + i\mu = P; \quad r_1 + ir_2 = \phi;$$
$$s_1 + is_2 = \psi; \quad c_1 + ic_2 = k. \tag{4}$$

Then we find the following equations of motion.

$$\left.\begin{aligned}
&A_1\ddot{\zeta} + F\ddot{\zeta}'' + i\omega(C_1\dot{\zeta} + F\dot{\zeta}'') = 0,\\
&A_1\ddot{\zeta}'' + F\ddot{\zeta} + i\omega\{F\dot{\zeta} + (2A_1 - C_1)\dot{\zeta}''\} + (C_1 - A_1)\omega^2\zeta'' + P = 0,\\
&A_0\ddot{\zeta}' + i\omega(2A_0 - C_0)\dot{\zeta}' + (C_0 - A_0)\omega^2\zeta'\\
&\quad + \frac{C}{2(1+\eta\alpha^5)}\, i\omega(\dot{\phi} + \eta\alpha^5\dot{\psi}) - \frac{C\omega^2}{2(1+\eta\alpha^5)}(\phi + \eta\alpha^5\psi) + (C - A)k - P = 0,\\
&\qquad \frac{C}{2(1+\eta\alpha^5)}\left\{\frac{g}{a}(H_3\phi - \eta\alpha^5 H_1\psi) - k + i\omega\dot{\zeta}' - \omega^2\zeta'\right\} = 0,\\
&\frac{C\eta\alpha^5}{2(1+\eta\alpha^5)}\left\{\frac{g}{a}(-H_1\phi + H_2\psi) - k + i\omega\dot{\zeta}' - \omega^2\zeta'\right\} + \frac{A_1 - \frac{1}{2}C_1}{C_1 - A_1}P = 0,
\end{aligned}\right\} \tag{5}$$

$$\zeta'' - \zeta' + \frac{A_1 - \frac{1}{2}C_1}{C_1 - A_1}\psi = 0. \tag{6}$$

If all the variables in these equations are proportional to $\exp(i\gamma t)$ we have

$$\gamma\{(A_1\gamma + C_1\omega)\zeta + F(\gamma + \omega)\zeta''\} = 0, \tag{7}$$

$$(\gamma + \omega)[\{A_1\gamma - (C_1 - A_1)\omega\}\zeta'' + F\gamma\zeta] - P = 0, \tag{8}$$

$$(\gamma + \omega)\{A_0\gamma - (C_0 - A_0)\omega\}\zeta' + \frac{C\omega(\gamma + \omega)}{2(1+\eta\alpha^5)}(\phi + \eta\alpha^5\psi) - (C - A)k + P = 0, \tag{9}$$

$$\frac{C}{2(1+\eta\alpha^5)}\left[\frac{g}{a}(H_3\phi - \eta\alpha^5 H_1\psi) - k - \omega(\gamma + \omega)\zeta'\right] = 0, \tag{10}$$

$$\frac{C\eta\alpha^5}{2(1+\eta\alpha^5)}\left[\frac{g}{a}(-H_1\phi + H_2\psi) - k - \omega(\gamma + \omega)\zeta'\right] + \frac{A_1 - \frac{1}{2}C_1}{C_1 - A_1}P = 0, \tag{11}$$

with (6) as before.

3. *Solution for the Eulerian period.*—The problem is, given the period of the free nutation, to estimate σ and hence μ. To make the whole mass, the radius of the core, and the whole moment of inertia right we take

$$\alpha = 0{\cdot}545, \quad \eta = 1{\cdot}923, \quad \rho(1 + \eta\alpha^3) = 5{\cdot}53.$$

We also take

$$a = 6{\cdot}37 \times 10^8 \text{ cm.}, \quad 2\pi/\omega = 23^{\text{h}}\,56^{\text{m}};$$

$$\frac{C - A}{C} = 0{\cdot}003273; \quad \frac{C_1}{C} = 0{\cdot}1287; \quad \frac{C_1 - A_1}{C_1} = 0{\cdot}0022; \quad \frac{\omega^2 a}{g} = 0{\cdot}00345;$$

$$\gamma/\omega = 1/440 = 0{\cdot}002272 \pm 0{\cdot}000030; \quad g\rho a = 2{\cdot}64 \times 10^{12} \text{ dynes/cm.}^2;$$

$$H_1 = 0{\cdot}458 + 24{\cdot}5\sigma; \quad H_2 = 1{\cdot}349 + 12{\cdot}6\sigma; \quad H_3 = 0{\cdot}542 + 6{\cdot}21\sigma.$$

The ellipticities of the core and the outer surface determine the precessional constant; here I have used the precessional constant and the external ellipticity to determine the ellipticity of the core and hence $(C_1 - A_1)/C_1$. With the simplified model that we are using, it is impossible to fit all the data; in particular the C_1/C given by it differs appreciably from Bullen's value 0·112.

The solution, with $k = 0$, was carried out by Mr R. A. Brooker at the Mathematical Laboratory, Cambridge. The result is

$$\sigma = 0{\cdot}722,$$

and with 4·22 for the density of the shell this gives

$$\mu = 1{\cdot}89 \times 10^{12} \text{ dynes/cm.}^2.$$

My adopted value, also used by Rosenhead, was $1{\cdot}695 \times 10^{12}$ dynes/cm.^{2}, based on the use of an average velocity of transverse waves. It was necessarily extremely rough and the value of η has needed a correction, but the result found here does at least show that with the correct allowance for fluidity of the core we can still fit the Eulerian period with a reasonable rigidity. According to Bullen, μ in the shell ranges from 0·6 to $3{\cdot}0 \times 10^{12}$ dynes/cm.^{2}.

My previous solution and Rosenhead's are not strictly comparable with the present one, because the values adopted for ρ and η were affected by a numerical mistake, corrected by van Iterson, and because both applied a correction to Love's number k to allow for the ocean. The latter change, though desirable, has not been made here. With the present data, but using my previous method, I find $k = 0{\cdot}291$, $\sigma = 0{\cdot}752$, $\mu = 1{\cdot}99 \times 10^{12}$ dynes/cm.^{2}.

I have reworked Mr Brooker's solution and get $\sigma = 0{\cdot}718$; his is probably the more accurate, but the difference is in any case well within what might be produced by permissible changes of $C_1 - A_1$ and γ/ω. The arithmetic can be considered thoroughly checked. The ratios of the variables are of interest:

$$\phi = 0{\cdot}515\psi; \quad \zeta' = 262{\cdot}9\psi; \quad \zeta'' = 36{\cdot}6\psi; \quad \zeta = -0{\cdot}9978\zeta''.$$

With a rigid shell we should have $\zeta'' = \zeta'$, and ζ/ζ'' would still be $-0{\cdot}9978$. If we take ζ' as a standard, so as to preserve the amplitude of the general rotation of the shell, ζ'' is only about $0{\cdot}14\zeta'$, and ζ is reduced in proportion. It appears that the changes of pressure in the core are such that bending of the shell removes most of the motion of the core boundary that ζ' would produce by itself, even with the high rigidity that we have found for the shell. This reduction accounts for the surprising change of the free period in comparison with the case of a rigid shell; with the values of C_1 and $C_1 - A_1$ that we have taken, but with a rigid shell, ω/γ would be 265 instead of 440. It also accounts for the fact that the results show little change from those found by neglecting the inertia of the core; for there is after all comparatively little motion in the core.

4. *The lunar nutation.*—For this we restore k in the equations 2.5 (6) to (11). It is convenient to take as a standard of comparison the values for a rigid Earth. We therefore put

$$\gamma = -\omega + n; \quad k = -\frac{n(C\omega - An)}{C - A}\beta; \quad \zeta' = (1 + p)\beta.$$

For a rigid Earth we should have $\zeta' = \beta$. For the principal lunar nutation

$$n/\omega = -1/6800.$$

With the rigidity of the shell indicated by the variation of latitude in the last section the solution is

$$p = -0{\cdot}00663; \quad \phi = 0{\cdot}000057_1\beta; \quad \psi = 0{\cdot}0000788\beta;$$

$$\zeta'' = 0{\cdot}9785\beta, \quad \zeta' = 0{\cdot}9934\beta, \quad \zeta = 0{\cdot}0715\zeta'.$$

As we should expect from the case of a rigid shell, the results in the last line differ greatly from those for the variation of latitude. The ratio ζ''/ζ' is now near 1 and ζ is much less than either. p with a rigid shell, with the values adopted for the core constants, would be $-0{\cdot}0092$, so that my previous conjecture that elasticity of the shell would reduce the effect of fluidity of the core is verified. On the other hand, elasticity in a completely solid Earth would have very little effect; if we drop all core terms in the equations of motion, replacing A_0, C_0 by A, C, we find easily that with the adopted rigidity p would be of order 10^{-5} and positive.

The value found for p in my previous paper for a rigid shell was $-0{\cdot}0060$. The larger value found here for a rigid shell is due to the adoption of a model with a larger moment of inertia and a smaller ellipticity for the core. The present values have been adopted so that the data used would be consistent with a uniform shell superposed on a uniform core; Bullen's density distribution is of course far more accurate, but if we keep the radius of the core, the mean density, and the total moment of inertia correct on the simplified hypothesis, we necessarily make C_1 wrong. An alternative would be to adopt Bullen's moments of inertia for the core and alter the radius of the core to make them consistent with the mean density. This also makes the mass of the shell wrong, and as α enters through several different powers it seemed most important to use its correct value. But such considerations can only show that the most we can expect with a model of Wiechert type is a result of the right order of magnitude. Perhaps a solution for Bullen's data can ultimately be found by numerical methods; the period of the variation of latitude and the amplitude of the nutation would then provide two comprehensive checks. It is not obvious what the direction of the changes would be. On account of the reduced density at the outside of the core the kinetic energy of the core would be less; this would tend to reduce the effect of fluidity of the core. On the other hand, the increased rigidity in the deeper parts of the shell would tend to reduce the correction for elasticity.

4.1. *Correction for the obliquity of the ecliptic.*—We have taken the oscillations to be small disturbances from the mean axis of rotation. This, in the long run, would be the normal to the ecliptic, and the mean obliquity is not small. A correction can be made for this. Over intervals of a century or so we can use a mean position of the Earth's axis as axis of reference. The work function for the principal nutation term is of the form

$$W = \kappa \sin 2\theta \cos(\phi - nt),$$

where θ, ϕ are polar coordinates of the axis with regard to axes in the ecliptic, and n has its previous meaning ($n/\omega = -1/6800$). Take the standard position to be θ_0, ϕ_0, and put

$$\lambda = \theta - \theta_0, \qquad \mu = \sin\theta_0(\phi - \phi_0).$$

Then we can put $\phi_0 = 0$ after differentiation and take

$$\delta W = 2\kappa \cos 2\theta_0 \cos nt \,.\, \lambda + 2\kappa \cos\theta_0 \sin nt \,.\, \mu.$$

In terms of our previous notation

$$\lambda + i\mu = e^{-i\omega t}(l' + im') = e^{-i\omega t}\zeta' = Z \text{ say,}$$

and we have worked out the theory when the generalized forces on l', m' are the real and imaginary parts of a multiple of $\exp i(-\omega + n)t$. Then those on λ, μ will be those of a multiple of $\exp(int)$. Now if the response to $\exp(int)$ for a rigid Earth is $Z = u(n)\exp(int)$, that for the composite Earth is $Z = (1+p)u(n)\exp(int)$. The response to $\exp(-int)$ will be $-(1-p)u(n)\exp(-int)$ so long as n/ω is small compared with $(C-A)/A$.

Now

$$\frac{\partial W}{\partial \lambda} + i\frac{\partial W}{\partial \mu} = \kappa\{(\cos 2\theta_0(e^{int} + e^{-int}) + \cos\theta_0(e^{int} - e^{-int})\}$$

$$= \kappa\{(\cos\theta_0 + \cos 2\theta_0)e^{int} - (\cos\theta_0 - \cos 2\theta_0)e^{-int}\}$$

and

$$Z = \kappa u(n)\{(1+p)(\cos\theta_0 + \cos 2\theta_0)e^{int} + (1-p)(\cos\theta_0 - \cos 2\theta_0)e^{-int}\}$$

$$= 2\kappa u(n)\{(\cos\theta_0 + p\cos 2\theta_0)\cos nt + i(\cos 2\theta_0 + p\cos\theta_0)\sin nt\}.$$

Hence λ is multiplied by $1 + p\sec\theta_0\cos 2\theta_0$, μ by $1 + p\sec 2\theta_0\cos\theta_0$. The constitution that we have adopted here makes p about $-0{\cdot}0066$; then with the actual value of θ_0 the nutation in latitude should be decreased by about 5 parts in 1000, that in longitude by about 9 parts in 1000.

Observations of the nutation have always been presented in a form adapted to the nutation in latitude. Sir H. Spencer Jones tells me that a separate estimation of that in longitude could be made from the data. The prediction that the two components should be reduced in different ratios is independent of the value of p and therefore of the special constitution used in the present discussion, and would be worth testing.

5. *Discussion.*—The result that with a rigid shell a fluid core would shorten the period of the free variation of latitude in comparison with that for a completely rigid Earth made it necessary to investigate whether the observed period can be reconciled with the existence of a fluid core by allowing for elasticity of the shell. This is now shown to be possible; the appropriate rigidity of the shell is not far from what had been found suitable in previous calculations. This is a consequence of the fact that even with a comparatively small flexibility of the shell much less disturbance is made in the core than by a rigid shell. Previous arguments, that with a core rigidity comparable with that of the shell the free period would be too short but that the observed period is consistent with a fluid core, therefore retain their validity.

The model adopted for discussion is of Wiechert type. With a rigid shell it makes the effect on the amplitude of the lunar nutation greater than with the constants adopted in the previous discussion, and this was already three times as large as was needed to remove the known discrepancy between the theoretical and calculated amplitudes. The effect of elasticity of the shell is found to introduce a factor of about 0·7, and the obliquity of the ecliptic a further factor of about 0·75 on the nutation in latitude; but even with these adjustments the effect remains too great, though the excess is possibly not greater than might be due to the simplifications made in the adopted model.

The distributions of density and rigidity are now sufficiently well known to permit a satisfactory comparison of both the free period and the nutation constant with theory, apart from the labour of calculation. It appears that this may not be prohibitive (in fact a numerical solution based on Knott's values was given long ago by Stoneley). The shell theory will still be statical and could be developed for arbitrary q and q_1 by relaxation methods.* An approximation to the kinetic energy of the core could be found by using the present forms of the displacement; by Hamilton's principle this should lead to a reasonably accurate result.

I remarked in my previous paper that variability of density in the core would not affect Helmholtz's equations so long as density is a function of pressure only. This is true, but my conclusion that Lamb's treatment would remain valid in these conditions is not. In the first place, the vorticity would vary with position, but is taken uniform in Lamb's theory. In the second, the relation between the moments of inertia A_1, C_1 and the ellipticity of the core boundary would not be true for a non-uniform fluid in equilibrium, for it is known from the theory of the figure of the Earth that the ellipticities of strata of equal density increase with distance from the centre.

The associated tidal deflection of the vertical may throw a systematic error into the measured nutation; I propose to examine this point later.

160 *Huntingdon Road,*
Cambridge :
1949 *November* 9.

* I understand from Dr W. D. Lambert that a solution has been made by Takenti, in Japan, based on Bullen's values for the density and rigidity. Core inertia was neglected, but it appears that the most laborious part of the work has already been done.

ON THE CONSTITUTION OF MARS

K. E. Bullen

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Summary

It is shown that the Earth model based on a compressibility-pressure hypothesis fits the theories that the density change across the boundary of the Earth's central core is purely a pressure phenomenon and that the Earth and Mars are of the same primitive composition. The fit is best if it is assumed that Mars has an *inner* core in the same proportion by mass as in the Earth and, correspondingly, that the material in the inner core of the Earth is chemically distinct from the material throughout the rest of the Earth below the crustal layers. This last pair of suggestions receives some support from a consideration of the moment of inertia and ellipticity of Mars.

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1. On the basis of an Earth model *, subsequently to be referred to as Model A, and assuming that the density variation in the Earth's deep interior is essentially determined by pressure rather than by changes in the materials present, Ramsey † found an appreciable discrepancy in fitting the planet Mars to the hypothesis that the terrestrial planets are of the same primitive composition. An independent calculation by the writer confirms this calculation of Ramsey's. The discrepancy is, however, much less than that found by Jeffreys ‡ on the hypothesis that the large change in density across the boundary of the Earth's central core is due to a change of material rather than to pressure. Ramsey has shown that the discrepancy can be removed by modifying Model A by including an allowance for gravitation of heavier elements towards the centre of the Earth.

In the present paper, it will be shown that the discrepancy is fully removed by using in place of Model A an Earth model §, to be subsequently referred to as Model B, constructed on the basis of a compressibility-pressure hypothesis. This result derives significance from the fact that the process of construction of Model B is entirely independent of the observations of radius and mass of Mars and of Venus, whereas the numerical allowances made by Ramsey are estimated from observational data for Mars.

2. As in a previous paper on Venus ||, the crustal layers of Mars will be taken as equivalent to a single layer of thickness 30 km. and density 2·8 g./cm.^{3}. This is necessarily arbitrary, but the arbitrariness is probably not a source of serious error.

On the data used by Ramsey, the mass M and radius R of Mars are given by

$$M=(6{\cdot}442\pm0{\cdot}013)\times10^{26}\,\text{g.};\qquad R=3396\pm8\,\text{km.}\qquad(1)$$

In a first calculation, the central values $M=6{\cdot}442\times10^{26}$ g. and $R=3396$ km. were used. The procedure followed was as in Paper II, except that the density-

* K. E. Bullen, *An Introduction to the Theory of Seismology*, Cambridge, p. 218, 1947.

† W. H. Ramsey, *M.N.*, **108**, 406, 1948.

‡ H. Jeffreys, *M.N., Geophys. Suppl.*, **4**, 62, 1937.

§ K. E. Bullen, *M.N., Geophys. Suppl.*, **6**, 50, 1950; this paper will be subsequently referred to as Paper I.

|| K. E. Bullen, *M.N.*, **109**, 457, 1949; this paper will be subsequently referred to as Paper II.

pressure values used below the crustal layers were those appropriate to Model B, as given in Table III of Paper I. The numerical integration was first carried out down to a distance of 800 km. from the centre of Mars, the yielded value of the mass inside the sphere of this radius being $0{\cdot}164 \times 10^{26}$ g., and the corresponding mean density, σ say, 7·7 g./cm.^{3}.

3. Suppose first that the inner core of the Earth is due primarily to pressure rather than to a change of material, so that Mars would have no inner core. Then if Mars and the Earth are of the same primitive composition, the value of σ found in Section 2 is too great by about 3·5 g./cm.^{3}.

The value of σ is, however, very sensitive to changes in the value of R given by (1) (*cf.* the case of Venus, Paper II), and R needs in fact to be increased only by 14 km. in order to bring about the needed reduction in σ.

Thus the hypotheses that

(*a*) for Mars $M = 6{\cdot}442 \times 10^{26}$ g., $R = 3410$ km.;
(*b*) the pressure-density figures for the Earth are as in Model B;
(*c*) the density is a pure function of the pressure throughout the Earth;
(*d*) Mars and the Earth are of the same primitive composition;

are totally compatible.

The corresponding distribution of the density ρ and pressure p inside Mars below the crustal layers is given in Table I in terms of the distance r from the centre and the depth d below the outer surface.

TABLE I

d	r	ρ	p
30 km.	3380 km.	3·30 g./cm.^{3}	$0{\cdot}003 \times 10^{12}$ dynes/cm.^{2}
211	3199	3·36	0·025
211	3199	3·87	0·025
500	2910	3·93	0·064
1000	2410	4·02	0·12
1500	1910	4·10	0·17
2000	1410	4·15	0·21
2500	910	4·19	0·24
3000	410	4·22	0·26
3410	0	4·23	0·26

The value $R = 3410$ km. is 6 km. outside the standard error given in (1). If the value taken for M were reduced to $6{\cdot}429 \times 10^{26}$ g., i.e. the lowest value in the range given in (1), the needed value of R would be reduced by $2\frac{1}{2}$ km. Thus the hypotheses (*b*), (*c*) and (*d*) when taken together give a result just outside the standard errors of the observational data (1). Thus the discrepancy referred to in Section 1 is almost but not quite removed by using Model B and assuming that Mars lacks an inner core.

4. The value of σ obtained in Section 2 suggests that instead of taking the hypothesis (*c*) of Section 3, we should consider the hypothesis:

> (*c'*) that the density is a pure function of the pressure inside the Earth between the crustal layers and the inner core, but that the inner core consists of material chemically distinct from the rest of the subcrustal material.

In the absence of knowledge of the composition of the layer F of the Earth, it will be assumed that the mass of this chemically distinct material is equal to

the mass of the region G plus half that of F; by Paper I, this mass is $1{\cdot}54 \times 10^{26}$ g. If Mars has a similar inner core in the same proportion by mass as in the Earth, the mass would be $0{\cdot}166 \times 10^{26}$ g.

The determination of the mean density of an inner core in Mars involves an extrapolation from the pressure of order $3{\cdot}5 \times 10^{12}$ dynes/cm.^{2} in the Earth's inner core to a pressure of order $0{\cdot}33 \times 10^{12}$ dynes/cm.^{2} that would obtain near the centre of Mars. In the absence of other knowledge, it will be assumed that the material of the inner core obeys the pressure-incompressibility formula

$$k = 2{\cdot}25 + 2{\cdot}86p + 0{\cdot}16p^2, \tag{2}$$

where the units of k and p are 10^{12} dynes/cm.^{2}, found in Paper I to hold approximately throughout most of the Earth outside the inner core.

Since $k = \rho\, dp/d\rho$ for a given material, we have from (2), ignoring possible thermodynamical complications,

$$\rho^{-1}\, d\rho = (2{\cdot}25 + 2{\cdot}86p + 0{\cdot}16p^2)^{-1}\, dp,$$

which yields on integration

$$C\rho^{2{\cdot}60} = \frac{0{\cdot}32p + 0{\cdot}26}{0{\cdot}32p + 5{\cdot}46}, \tag{3}$$

where C is constant. Taking $\rho = 16$ when $p = 3{\cdot}5$, as given approximately in the inner core of the Earth, we find that $C = 10^{-4} \times 1{\cdot}55$. For $p = 0{\cdot}33$, we then find from (4) that $\rho = 10{\cdot}2$ g./cm.^{3}, which is accordingly the best figure to adopt for the mean density of the inner core of Mars.

The radius of the inner core of Mars is deduced from the above figures for the mass and mean density to be 730 km. If we take as data

$$M = 6{\cdot}442 \times 10^{26}\text{ g.}; \qquad R = 3391\text{ km.}, \tag{4}$$

and follow the process described in Section 2, the mass inside a sphere of radius 730 km. in Mars is found to be $0{\cdot}166 \times 10^{26}$ g. Thus the data (4) are compatible with (*b*), (*c'*) and (*d*). Moreover the data (4) are well within the standard errors in (1). The discrepancy is therefore quite removed if Mars has an inner core like that in the Earth.

Comparing this with the result in Section 3, we see that one result of the present calculations is to favour slightly the possibility that the inner core in the Earth consists of a material or mixture of materials that is chemically distinct from the rest of the Earth. If, in the formula (3), we put $p = 0$, we find $\rho = 9{\cdot}0$ g./cm.^{3}, which is a little greater than the densities of iron and nickel at zero pressure. Thus the present results are compatible with the existence of a large proportion of nickel-iron in the inner cores of the planets, together with some additional material of higher density. This goes some distance towards removing the objection to the new pressure-density theory that it allows no room for concentrated nickel-iron in the central cores. (Of course, not too much weight should be attached to the precise figure of 9·0 g./cm.^{3} because of the very considerable extrapolation involved.)

The density-pressure distribution inside Mars given by the revised calculations is as in Table II. (No provision is made for a transition layer corresponding to the layer F in the Earth; such a layer might not be present in Mars, since there is no layer in Mars corresponding to the region E of the Earth.)

TABLE II

d	r	ρ	p
30 km.	3361 km.	3·30 g./cm.^{3}	0·003 × 10^{12} dynes/cm.^{2}
208	3183	3·36	0·025
208	3183	3·87	0·025
500	2891	3·93	0·064
1000	2391	4·03	0·13
1500	1891	4·11	0·18
2000	1391	4·17	0·23
2500	891	4·23	0·26
2661	730	4·25	0·28
2661	730	10·1	0·28
3391	0	10·3	0·35

5. It is of further interest to compare estimates of the ellipticity e of Mars with values of the moment of inertia I determined from various hypotheses on the density distribution in Mars.

Struve's estimate of e is 1/192 with a probable uncertainty of 2 to 3 per cent.

The values of I/MR^2 yielded for Mars by using (i) the Earth Model A, assuming that Mars has no inner core, (ii) the Earth Model A, assuming Mars has an inner core, (iii) the density distribution of Table I, (iv) the density distribution of Table II, are 0·389, 0·380, 0·391, 0·382, respectively. Assuming hydrostatic conditions throughout Mars, the Radau-Darwin equations

$$1-3I/2MR^2=2(1+\eta)^{\frac{1}{2}}/5, \qquad \eta=5m/2e-2, \tag{5}$$

where m denotes the ratio of the centrifugal force at the equator to the gravitational attraction, are relevant; and give for e the values 1/183, 1/189, 1/181, 1/188 corresponding to (i), (ii), (iii), (iv), respectively.

The assumption of hydrostatic conditions inside the Earth leads to an estimate of the Earth's ellipticity which agrees with independent estimates * within about 0·2 per cent. It is to be expected that the relative volume in Mars in which the fundamental strength is not exceeded by the stresses present will be appreciably greater than in the case of the Earth, and therefore, as pointed out by Ramsey and earlier by Jeffreys †, estimates of e on the hydrostatic theory may have appreciably greater errors in the case of Mars than in the case of the Earth. But the smallness of the discrepancy in the case of the Earth suggests that the estimates 1/189 and 1/188 correspond to much more likely density distributions in Mars than the estimates 1/183 and 1/181, i.e. that (ii) and (iv) are much more probable than (i) and (iii). Both the values 1/189 and 1/188 lie within the margin of uncertainty of Struve's result.

Investigation of e on the basis of either the Earth Model A or B therefore gives added support to the evidence in Section 4 that Mars has an inner core and that the inner cores of the planets are chemically distinct from the outer parts. And the earlier argument shows that the Model B must be preferred to A in fitting together the hypotheses (c) or (c') and (d).

Ramsey does not give a revised estimate of e based on his method of removing the discrepancy referred to in Section 1. It is likely that an *ad hoc* adjustment in

*E. C. Bullard, *M.N., Geophys. Suppl.*, **5**, 186, 1948. H. Jeffreys, *M.N., Geophys. Suppl.*, **5**, 219, 1948.

† H. Jeffreys, *M.N., Geophys. Suppl.*, **4**, 62, 1937.

his procedure would lead to a value of e within 2 per cent of 1/192 without the need for assuming an inner core on Mars, and to this extent caution must be used in following the suggestion of the last paragraph. The Model B has, however, the advantage pointed out in Section 1 that no *ad hoc* adjustments are involved in fitting it to the planets. If it transpires that (c) or (c') and (d) are sustained, it will be desirable to seek tests to discriminate between Model B and Ramsey's modification of Model A. Suggestions already made at the end of Paper I may in due course achieve this result.

In so far as the Model B with the inner core assumed to be chemically distinct leads, on the above discussion, to results that are inside the standard errors of all relevant astronomical data, this model and the compressibility-pressure hypothesis which determines it have added weight as working hypotheses.

University of Sydney,
N.S.W.:
1949 *October* 20.

THE CALCULATION OF THE CONVERGENT OF A MOVING CLUSTER OF SMALL APPARENT DIAMETER

R. M. Petrie

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Summary

The methods of Charlier and Bohlin, for calculating the convergent point of a moving cluster, are discussed and it is shown that a number of different results may be expected from the various possible arrangements of the equations. Examples are given showing that the differences are considerable for groups occupying a limited area on the sky. A method of differential corrections is proposed, more directly related to the observed proper motions, and giving a less ambiguous result for condensed clusters of the Pleiades type.

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1. The calculation of the convergent point of a moving cluster from the proper motions of the individual stars usually follows the methods of Charlier or of Bohlin.* Since the two methods merely represent two different arrangements of the same relation, we expect them to give the same answer, and this is, in fact, very nearly true for some clusters. There are important examples, however, where the two methods fail to give the same convergent; this situation obtains when the stars are contained in a small area on the sky and have nearly equal proper motions. Thus the question arises as to which convergent should be adopted. The purpose of this note is to demonstrate the variation encountered in practice in the derived convergents and to propose a method more directly related to the observed motions and yielding an unambiguous result.

2. In Charlier's solution we have an equation of condition for each star, of the form

$$aX+bY+cZ=0, \tag{1}$$

where X, Y and Z are the equatorial linear components of the space motion, V, and a, b and c are functions of the coordinates and proper motions. They are defined by

$$\begin{aligned} a &= \eta \sin\alpha - \xi \cos\alpha \sin\delta, \\ b &= -\eta \cos\alpha - \xi \sin\alpha \sin\delta, \\ c &= \xi \cos\delta, \end{aligned}$$

where $\xi = \mu_\alpha \cos\delta$ and $\eta = \mu_\delta$. A least-squares solution of (1) for all the stars in the cluster gives the ratios X/Y, Z/Y, etc., from which the coordinates, (A, D), of the convergent point are found, since

$$X = V\cos A\cos D, \qquad Y = V\sin A\cos D, \qquad Z = V\sin D.$$

Bohlin's equation is written:

$$lx+my+cz=0, \tag{2}$$

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* For a derivation of the formulae, and a demonstration that the equations are fundamentally the same in the two methods, see W. M. Smart, *Stellar Dynamics*, Chap. IX, Cambridge, 1938.

where l, m, n are the direction cosines of the northern pole of the great circle through the star in the direction of the total proper motion. If (a, d) are the right ascension and declination, respectively, of that pole, then

$$l = \cos a \cos d, \qquad m = \sin a \cos d, \qquad n = \sin d,$$

where

$$\sin d = \cos\delta \sin\theta,$$
$$\cos d \sin(a-\alpha) = -\cos\theta,$$
$$\cos d \cos(a-\alpha) = -\sin\theta \sin\delta,$$

and θ is the position angle of the total proper motion defined by

$$\tan\theta = \frac{\mu_\alpha \cos\delta}{\mu_\delta}.$$

The unknowns x and y are given by

$$x = \cot D \cos A, \qquad y = \cot D \sin A.$$

We see that $x = X/Z$ and $y = Y/Z$ and it may be shown that

$$l/a = m/b = n/c = 1/\mu.$$

Bohlin's equation then may be written as

$$\frac{aX}{\mu Z} + \frac{bY}{\mu Z} + \frac{c}{\mu} = 0, \tag{3}$$

and is therefore the same as Charlier's equation except that the coefficients bear the ratio $1/\mu$, which is of course different for each individual observation equation.

It may be remarked that other arrangements are possible; for example, we may write (2) as

$$x \cos a + y \sin a + \tan d = 0, \tag{4}$$

and then the coefficients are related to those of Charlier by the factor $(1/\mu)\sec d$. Professor Smart has pointed out* that the effect of the factors described above is to weight the equations in one solution relative to those in the other; hence, if the errors of observation are not negligible, the results will not agree in the different solutions.

There appears to be another difference between solutions, which may be illustrated as follows. Solving (1) by least-squares gives values of X/Y and Z/Y such that $\Sigma p(b-b')^2$ is minimized, where b is computed directly as defined above and b' is found by substituting the derived values of X/Y and Z/Y in $-\{a(X/Y)+c(Z/Y)\}$. Let us now solve (1) for the ratios X/Y and Y/Z. The condition upon the solution now is that $\Sigma p(c-c')^2$ shall be a minimum. The various possible arrangements thus place different conditions upon the unknowns and we do not, therefore, expect the answers to be identical.

3. When the stars composing a moving cluster are well separated in the sky and the proper motions exhibit a reasonable amount of convergence, the solution is stable and the deduced convergent point is not greatly dependent upon the particular form of solution chosen. When the stars are located close together on the sky, however, and have nearly equal proper motions, the solution becomes nearly indeterminate and the convergent point varies considerably, depending upon the relative weighting factors and the quantities which are minimized.

* Private communication.

The Taurus cluster provides a good example of the first case. In Table I are given the convergents calculated from the three arrangements of equations (1) and (2). The effect of the weighting of Bohlin's equation relative to that of Charlier may be seen by comparing the convergents in corresponding lines in the two sets; the effect of minimizing different quantities in the solutions may be seen by intercomparison of the three entries in a set. Evidently all forms of solution give closely the same convergent point for the Taurus cluster, although the differences between the second and third values, in each set, are a little greater than one would expect from the individual probable errors. It should be mentioned that, in order to reduce the computational labour, the stars in the Taurus cluster were grouped into nine mean points, since the purpose here is to illustrate the influence of the arrangement of the solution. The coordinates of the convergent point in Table I are therefore not directly comparable with the values $A = 91°{\cdot}8$, $D = +8°{\cdot}2$ derived by Smart* from the individual treatment of seventy-two stars.

The eleven nucleus stars of the Ursa Major cluster, on the other hand, give convergents depending very strongly upon the form of the equation of condition. The results for this cluster are shown in Table II, from which the divergences are apparent. In such a situation one despairs of obtaining much useful information by the conventional methods of treating the proper motions.

4. The question arises which, of the possible convergents, is the proper one to adopt. It is not easy to select one as being the best, since different conditions are imposed upon the unkowns in the various cases and the quantities minimized are rather involved functions of the coordinates and proper motions. Remembering that the convergent point is determined by the directions of the total proper motions, it is proposed that the "best" convergent be defined as that which agrees best with the observed position angles of the proper motions of the cluster stars, the agreement to be measured by the sum of the weighted squares of the residuals. In the case of the Ursa Major nucleus stars none of the conventional solutions fulfil this condition.

We proceed to find an explicit relation between the direction of the total proper motion and the coordinates of the convergent point. The proposed method is to find differential corrections to the right ascension and declination of an approximate convergent point under the condition stated above. Let the position of a star belonging to the cluster be (α, δ) and that of the initially chosen convergent be (A, D) and let θ be the position angle of the total proper motion and λ the angular distance from the star to the convergent. Then we have:

$$\sin\theta = \frac{\sin(A-\alpha)\cos D}{\sin\lambda},$$

$$\cos\theta = \frac{\sin D - \sin\delta\cos\lambda}{\cos\delta\sin\lambda}$$

and

$$\cos\lambda = \sin\delta\sin D + \cos\delta\cos D\cos(A-\alpha),$$

all of which are readily derived from the spherical triangle Pole-Star-Convergent. Eliminating λ from the above we may express θ explicitly in terms of the

* W. M. Smart, *M.N.*, **99**, 168, 1939.

coordinates of star and convergent as follows:

$$\cot\theta = \cos\delta \operatorname{cosec}(A-\alpha)\tan D - \sin\delta\cot(A-\alpha). \tag{5}$$

Differentiating (5) we get

$$\sin^2\theta\operatorname{cosec}(A-\alpha)\{\cos\delta\cot(A-\alpha)\tan D - \sin\delta\operatorname{cosec}(A-\alpha)\}dA - \sin^2\theta\operatorname{cosec}(A-\alpha)\cos\delta\sec^2 D\,dD = d\theta = O-C. \tag{6}$$

Equation (6) is our desired equation of condition giving, when solved by least-squares, corrections dA and dD to an assumed convergent (A, D) so as to minimize $\Sigma p(\theta_0-\theta_c)^2$. Solutions for the Taurus and Ursa Major convergents have been made by the differential method, with the results shown in the last lines in Tables I and II. The convergent thus found for the Taurus cluster agrees quite well with those derived by the methods of Charlier and Bohlin, but in the case of the nucleus stars of the Ursa Major cluster there is little agreement.

TABLE I

The convergent of the Taurus cluster

Method	Unknowns	A (R.A.)	D (Decl.)
Charlier	$\frac{X}{Y}, \frac{Z}{Y}$	$93^\circ\cdot1\pm0^\circ\cdot9$	$+7^\circ\cdot8\pm0^\circ\cdot3$
	$\frac{X}{Z}, \frac{Y}{Z}$	$92\cdot6\pm1\cdot6$	$+8\cdot0\pm0\cdot3$
	$\frac{Y}{X}, \frac{Z}{X}$	$97\cdot0\pm2\cdot0$	$+6\cdot4\pm3\cdot1$
Bohlin	$\frac{X}{Y}, \frac{Z}{Y}$	$93\cdot9\pm1\cdot0$	$+7\cdot5\pm0\cdot4$
	$\frac{X}{Z}, \frac{Y}{Z}$	$93\cdot1\pm1\cdot2$	$+7\cdot8\pm0\cdot4$
	$\frac{Y}{X}, \frac{Z}{X}$	$97\cdot7\pm1\cdot9$	$+6\cdot1\pm2\cdot3$
Differential	dA, dD	$94\cdot6\pm1\cdot0$	$+7\cdot4\pm0\cdot4$

The application of (6) to the Ursa Major stars giving the convergent $A=298^\circ$, $D=-31^\circ\cdot6$ has been found to be quite satisfactory. The writer has shown elsewhere* that this new convergent reconciles the observed proper motions, parallaxes, and recently revised radial velocities, with the assumption of community of motion, whereas the convergents calculated by the other methods fail to do so. This lends support to the method proposed here and inclines one to suggest that it may be useful in providing improved knowledge of the space motions of a number of moving clusters of the Pleiades type.

* *Publ. Dom. Ap. Obs.*, **8**, No. 2 (in press).

TABLE II

The convergent of the Ursa Major cluster nucleus

Method	Unknowns	A (R.A.)	D (Decl.)
Charlier	$\frac{X}{Y}, \frac{Z}{Y}$	$288^\circ \pm 10^\circ$	$-20^\circ \pm 15^\circ$
	$\frac{X}{Z}, \frac{Y}{Z}$	$343^\circ \pm 13^\circ$	$-55^\circ \pm 2^\circ$
	$\frac{Y}{X}, \frac{Z}{X}$	$339^\circ \pm 11^\circ$	$-54^\circ \pm 2^\circ$
Bohlin	$\frac{X}{Y}, \frac{Z}{Y}$	$275^\circ \pm 2^\circ$	$-1^\circ \pm 2^\circ$
	$\frac{X}{Z}, \frac{Y}{Z}$	$331^\circ \pm 12^\circ$	$-52^\circ \pm 3^\circ$
	$\frac{Y}{X}, \frac{Z}{X}$	$228^\circ \pm 3^\circ$	$-17^\circ \pm 3^\circ$
Differential	dA, dD	$298^\circ \pm 6^\circ$	$-32^\circ \pm 6^\circ$

The writer wishes to thank Professor W. M. Smart for his kind interest and helpful advice in preparing this note. Dr B. N. Moyls, of the University of British Columbia, gave valuable assistance in discussing some of the problems involved. Thanks are due also to Mrs J. Williams of the Observatory staff who performed much of the computational work required for Tables I and II.

The Dominion Astrophysical Observatory,
Victoria, B.C. :
1949 *March.*

THE QUANTAL THEORY OF CONTINUOUS ABSORPTION OF RADIATION BY VARIOUS ATOMS IN THEIR GROUND STATES

II. Further Calculations on Oxygen, Nitrogen and Carbon *

D. R. Bates and M. J. Seaton

(Received 1949 November 19)

Summary

The absorption cross-sections of atomic oxygen, nitrogen and carbon are calculated, the first with rather more precision than the other two. Exchange wave functions and both the dipole moment and dipole velocity formulae are used. At the spectral heads the absorption cross-sections, in units 10^{-18} cm.^{2}, are as follows : *O*, 2·6; *N*, 9; *C*, 11.

1. *Introduction.*—Quantal calculations have now been carried out on the continuous radiative absorption cross-sections of a considerable number of different systems.† Information is available on the general trend exhibited in at least the earlier part of the Periodic Table; but much remains to be done on increasing the accuracy of the calculations on individual systems. In some cases the cross-section is so sensitive to details of the wave functions involved that further numerical work is scarcely profitable; for instance, it is not yet possible to calculate reliably even the order of magnitude of the cross-section of atomic potassium. However, in other cases great sensitivity does not arise and it is worth attempting to improve the precision of the calculations already performed. The effect of including allowance for electron exchange is investigated in the present paper, and in addition a comparison is made of the results obtained, using the alternative dipole moment and dipole velocity forms for the transition matrix element. Atomic oxygen is selected as the system for detailed study, partly because it is a typical representative of an important group, but mainly because accurate knowledge of its absorption cross-section is required in the theory of the formation of the ionized layers of the Earth's upper atmosphere‡: atomic nitrogen and carbon are also treated, but less fully.

2. *Procedure and results*

2.1. *The general formulae.*—As the method of calculating the absorption cross-section A of an atom is well known§, it need not be described in detail here. On substituting numerically for the various fundamental constants involved, the

* Paper I, D. R. Bates, *M.N.*, **100**, 25, 1939.

† The earlier work has been reviewed by D. R. Bates, *M.N.*, **106**, 432, 1946. Recently Su-Shu Huang, *Ap. J.*, **108**, 354, 1948, has investigated *He* (1^1S, 2^1S and 2^3S); L. C. Green, *Ap. J.*, **109**, 289, 1949, Ca^+ (4^2S, 4^2P and 3^2D); L. Biermann and K. Lübeck, *Zeits. f. Astrophysik*, **26**, 145, 1949, Mg^+ (3^2S, 3^2P, 3^2D and 4^2F), Si^+ (4^2S, 3^2P and 4^2P); and L. Biermann, *Nach. Akad. Wiss. Göttingen, Math.-Phys. Klasse*, 1947, Mg^+ ($3s$, $3p$, $3d$, $4f$), Si^+ ($4s$, $3p$, $4p$, $3d$, $4f$).

‡ *Cf.* D. R. Bates and M. J. Seaton, *Proc. Phys. Soc.* (in course of publication).

§ *Cf.* D. R. Bates, *loc. cit.*, 1946.

standard quantal formula, based on the dipole moment transition matrix element, reduces to

$$A = 8{\cdot}5_5 \times 10^{-19}(I+\epsilon)\{\mathscr{C}_p \epsilon^{\frac{1}{2}}(C_{l-1}|\int_0^\infty P(n,l|r)P(\epsilon,l-1|r)r\,dr|^2 + C_{l+1}|\int_0^\infty P(n,l|r)P(\epsilon,l+1|r)r\,dr|^2)\}\ \text{cm.}^2, \quad (1)$$

I being the ionization potential and ϵ the energy of ejection (both in atomic units, 13·53 e.V.), $\mathscr{C}_p$ being a factor to allow for the distortion of the wave functions of the passive electrons by the transition, C_{l-1} and C_{l+1} being coefficients arising from the integration over the angular coordinates, and $P(n,l|r)/r$ and $P(\epsilon,l-1$, or $l+1|r)/r$ being the normalized radial wave functions of the initial and final states of the active electron, the first parameter denoting the principal quantum number or energy and the second the azimuthal quantum number. Chandrasekhar* has pointed out that increased accuracy can sometimes be achieved by using the dipole velocity transition matrix element. With this

$$\left|\int_0^\infty P(n,l|r)P(\epsilon,l-1|r)r\,dr\right| \quad (2)$$

in (1) is replaced by

$$\frac{2}{(I+\epsilon)}\left|\int_0^\infty \left(\frac{l}{r}P(\epsilon,l-1|r) - P'(\epsilon,l-1|r)\right)P(n,l|r)\,dr\right|, \quad (3)$$

and

$$\left|\int_0^\infty P(n,l|r)P(\epsilon,l+1|r)r\,dr\right| \quad (4)$$

by

$$\frac{2}{(I+\epsilon)}\left|\int_0^\infty \left(\frac{l+1}{r}P(\epsilon,l+1|r) + P'(\epsilon,l+1|r)\right)P(n,l|r)\,dr\right|, \quad (5)$$

the primes indicating differentiation with respect to r. The advantage of the dipole velocity formula is that it does not demand such precise knowledge of the wave functions at large radial distances as does the dipole moment formula; a defect is the introduction of derivatives of the wave functions.

2.2. *Oxygen.*—The wave function of the ground state of oxygen was computed by Hartree and Black † in 1933 using the self-consistent field model. This was the only approximation available when the original calculations (*cf.* I) on the absorption cross-section were being done. More recently Hartree, Hartree and Swirles ‡ have greatly refined the earlier work and derived wave functions corrected for the effect of exchange.

The Fock equations for the s and d radial wave functions of the ejected electron can readily be set up, using the method developed by Hartree and Hartree.§ They are as follows:

$$\left\{\frac{d^2}{dr^2} + 2v + \epsilon\right\}P(\epsilon,s|r) = \left(\eta_1 - \frac{2Y_0(1s,\epsilon s)}{r}\right)P(1s|r) + \left(\eta_2 - \frac{2Y_0(2s,\epsilon s)}{r}\right)P(2s|r) + \frac{2}{3}\frac{Y_1(2p,\epsilon s)}{r}P(2p|r) \quad (6)$$

* S. Chandrasekhar, *Ap. J.*, **102**, 223, 1945.
† D. R. Hartree and M. M. Black, *Proc. Roy. Soc.* A, **139**, 311, 1933.
‡ D. R. Hartree, W. Hartree and B. Swirles, *Phil. Trans. Roy. Soc.* A, **238**, 229, 1939.
§ D. R. Hartree and W. Hartree, *Proc. Roy. Soc.* A, **154**, 588, 1936.

and

$$\left\{\frac{d^2}{dr^2}+2v-\frac{6}{r^2}+\epsilon\right\}P(\epsilon,d|r)=-\frac{2}{5}\frac{Y_2(1s,\epsilon d)}{r}P(1s,|r)-\frac{2}{5}\frac{Y_2(2s,\epsilon d)}{r}P(2s|r)$$
$$+\left(\frac{4}{15}\frac{Y_1(2p,\epsilon d)}{r}+\frac{42}{245}\frac{Y_3(2p,\epsilon d)}{r}\right)P(2p|r), \quad (7)$$

where

$$Y_k(nl,\epsilon l')=\int_0^r P(n,l|r')P(\epsilon,l'|r')\left(\frac{r'}{r}\right)^k dr'+\int_r^\infty P(n,l|r')P(\epsilon,l'|r')\left(\frac{r}{r'}\right)^{k+1}dr', \quad (8)$$

v is the usual static potential of the core, and η_1 and η_2 are the adjustable orthogonality parameters. These equations were solved by numerical methods for zero ϵ, firstly neglecting, and secondly including the exchange terms on the right-hand side. $P(1s|r)$, $P(2s|r)$, $P(2p|r)$ (which, of course, relate to the ion) and the derived function v, were taken from the paper by Hartree, Hartree and Swirles.

The normalization of the continuous wave functions consists essentially in determining their asymptotic amplitudes. This can be done by fitting them to the corresponding Coulomb functions at sufficiently large radial distances.* Unfortunately, these functions have not been adequately tabulated †, and the task of computing them in individual cases is tedious. An alternative method, based on the Jeffreys' approximation, has been proposed by B. Strömgren ‡, but it does not appear to have been much used. We applied it in the present problem and found it very convenient. Beyond the region where the exchange terms are important, equations (6) and (7) are of the form

$$d^2P/dr^2+\lambda P=0. \quad (9)$$

As can readily be verified, this is satisfied by

$$P=Cz^{-\frac{1}{2}}\sin(\phi(r)), \quad (10)$$

where C is a constant and z, which represents $d\phi/dr$, is given by

$$z^2=\lambda+z^{\frac{1}{2}}\{d^2(z^{-\frac{1}{2}})/dr^2\}. \quad (11)$$

This equation can be solved by iteration; the convergence is extremely rapid—indeed it is usually only necessary to replace z in the second term by $\lambda^{\frac{1}{2}}$. As r tends to infinity, λ clearly tends to ϵ, so that the asymptotic amplitude of (10) is simply $C\epsilon^{-\frac{1}{4}}$. The determination of the multiplying constant C involved in any particular solution of (9) can be carried out rapidly. Choosing two radial distances r_1 and r_2 and introducing

$$\alpha=\tfrac{1}{2}\int_{r_1}^{r_2} z\,dr, \quad (12)$$

$$a_1=z_1^{\frac{1}{2}}P(r_1) \quad (13)$$

and

$$a_2=z_2^{\frac{1}{2}}P(r_2), \quad (14)$$

* J. Hargreaves, *Proc. Camb. Phil. Soc.*, **25**, 75, 1929.

† The only table of which we are aware is that by D. R. Bates and H. S. W. Massey, *Proc. Roy. Soc.* A, **177**, 329, 1941, for the case $l=1$; $\epsilon=0$ (0·025), 0·1; $r=2(0·2)\,4(0·5)\,10$. Bessel functions can, of course, always be used for $\epsilon=0$.

‡ *Cf.* M. Rudkjöbing, *Publ. Kbh. Obs.*, **18**, 1, 1941.

it can be shown by elementary analysis that

$$C = \tfrac{1}{2}\{(a_1+a_2)^2 \sec^2\alpha + (a_1-a_2)^2 \operatorname{cosec}^2\alpha\}^{\frac{1}{2}}. \qquad (15)$$

Since z is known from (11), this can easily be evaluated. As usual the functions finally used in the integrals in (1), (3) and (5) were adjusted to have $\epsilon^{-\frac{1}{2}}$ for their asymptotic amplitude.

The factor $\mathscr{C}_p$ equals

$$\prod_{n,\,l\,(\text{all passive electrons})} \left| \int_0^\infty P^i(n,l|r) P^f(n,l|r)\, dr \right|^2, \qquad (16)$$

the superscript i indicating a wave function relating to the initial atom and the superscript f indicating one relating to the final ion. It was computed using the P's published by Hartree, Hartree and Swirles and was found to be 0·94. For I the experimental value, 13·55 e.V.*, was adapted in preference to any theoretical estimate.

TABLE I

Continuous radiative absorption cross-section of atomic oxygen at spectral head

[A in units 10^{-18} cm.^{2}]

Approximation		Contribution from $p \to s$ transitions	Contribution from $p \to d$ transitions	Total
(*a*) *Dipole moment formula*				
$P(2p\|r)$	$P(\epsilon, s \text{ or } d,\|r)$			
(i) non-exchange	non-exchange	0·5·	3·9.	4·4†
(ii) non-exchange	exchange	0·7.	3·8·	4·5·
(iii) exchange	non-exchange	0·2·	1·8·	2·1
(iv) exchange	exchange	0·5	1·8·	2·3·
(*b*) *Dipole velocity formula*				
$P(2p\|r)$	$P(\epsilon, s \text{ or } d,\|r)$			
(i) non-exchange	non-exchange	0·6	3·0.	3·6.
(ii) non-exchange	exchange	0·4	2·9	3·3
(iii) exchange	non-exchange	0·6·	2·3	2·9·
(iv) exchange	exchange	0·5·	2·3.	2·8.

Recommended value 2·6

In order to examine the effect of the various approximations the absorption cross-section at the spectral head ($\epsilon = 0$) was calculated from both the dipole moment and dipole velocity formulae and for all combinations of the non-exchange and exchange wave functions. The results obtained are given in Table I, the contributions from $p \to s$ and $p \to d$ transitions being displayed separately. Attention may be drawn to the following points.

(1) The contribution from $p \to s$ transitions is considerably smaller than that from $p \to d$ transitions. This feature is very fortunate, as owing to cancellation of the positive and negative parts of the integrals involved the former contribution is more sensitive to details of the wave functions than is the latter. The difference in the reliability of the two is probably greater than the range of the figures in

* J. C. Boyce, *Rev. Mod. Phys.*, **13**, 1, 1941. As stated earlier I in formula (1) and expressions (3) and (5) are in atomic units.

† This is in agreement with a value given earlier (*cf.* D. R. Bates, *loc. cit.*, 1946).

Table I suggests, for, in any case in which serious cancellation occurs, there is always the potentiality of gross error unless the wave functions employed are extremely accurate. A useful measure of the degree of cancellation is got by multiplying the integral concerned by a suitable factor so that it can be written in the form $(1-D)$, where D is the lesser of the two parts of opposite sign. For $p \rightarrow s$ transitions the value of $(1-D)$ in one instance ((*a*, iii) of Table I) is as small as 0·63*, whereas for $p \rightarrow d$ transitions it is never appreciably below unity.

(2) Inclusion of allowance for exchange in the wave function of the ejected electron only affects the total cross-section slightly. The reason is that the centrifugal potential, $-6/r^2$, so dominates equation (7) that the *d*-component (which, as has just been mentioned, yields the major contribution) is scarcely altered by the additional terms. In contrast, exchange causes the wave function of the bound electron to be much more compact and consequently modifies the cross-section significantly.

(3) Comparison of results derived from the dipole moment and dipole velocity formulae reveals, as would be expected, that the former are in a more scattered group than are the latter; thus the ratios of the maximum to the minimum cross-sections in each group are 2·2 and 1·3 respectively.

Even with exchange fully included the two formulae give appreciably different cross-sections† (*cf.* Table I (*a*, iv) and (*b*, iv)). This is doubtless partly due to defects in the assumed wave functions of the active electron. However, a discrepancy would be anticipated with the most accurate possible wave functions of the type used, for these satisfy neither of the conditions for the basic formulae to be equivalent: namely that

(*a*) the wave function of the complete system satisfies the Schrödinger equation (and so cannot be represented in a form in which the coordinates of the individual electrons are separated) or

(*b*) the active electron moves in a simple potential field which is unaltered by the transition.

Unfortunately for a complex atom like oxygen it is scarcely feasible at present to improve upon the Fock model. In his very careful work on helium Su-Shu Huang‡ employed elaborate variational wave functions for the bound states. At the spectral head the ratios of the cross-sections obtained using the dipole moment formula to those obtained using the dipole velocity formula are 0·96, 1·12 and 0·96 for *He*, 1^1S, 2^3S and 2^1S respectively. The corresponding ratio for *O*, 2^3P is 1·23. Though the helium ratios are nearer unity than the oxygen ratio, they also differ appreciably from it. Clearly highly refined wave functions are necessary to ensure that the two formulae yield the same result. It is not easy to decide how to weigh the two different cross-sections obtained in practice. However, the most obvious procedure is to adopt a simple mean. For oxygen this gives a value of $2{\cdot}6 \times 10^{-18}$ cm.^{2}. The accuracy of this figure is difficult to estimate theoretically. To make a precise measurement on such a substance as

* For comparison it may be noted that in the case of potassium (D. R. Bates, *Proc. Roy. Soc.* A, **188**, 350, 1947), $(1-D)$ was found to range between 0·388 and 0·015, according to the approximation used for the wave functions; the cancellation is thus much more severe than for oxygen.

† An analogous effect arises in the calculation of the oscillator strengths of spectral lines (*cf.* D. R. Bates and A. Damgaard, *Phil. Trans. Roy. Soc.* A, **242**, 101, 1949). If the theoretical value of I were used in expressions (3) and (5) (as should strictly be done when comparing the cross-sections), the difference would be slightly enhanced.

‡ *Loc. cit.*, 1948.

atomic oxygen is likely to prove a formidable task. It should, however, be possible to investigate neon experimentally. If this were done (and it is understood that the investigation is being attempted in several laboratories) a comparison of the measured cross-section with that calculated would give an indirect but very useful indication of the reliability of theory for the case of oxygen. Some time ago Bates (I) carried out calculations on neon, using the dipole moment formula and non-exchange wave functions. Judging from the present work these are likely to have given an overestimation of the cross-section.* Fock wave functions for the system are not available at present, but it is hoped that they will become available in the near future. It is then planned to evaluate the cross-section of neon to the same degree of approximation as has been done for oxygen.

2.3. *Nitrogen and carbon.*—The earlier calculations on nitrogen and carbon employed the bound wave functions computed by Brown, Bartlett and Dunn, and by Torrance † respectively. In the derivation of these the effect of exchange was not taken into account. It has, however, been allowed for in the more recent work of Jucys (carbon) and Hartree and Hartree (nitrogen)‡ whose results are adopted here. As regards the ejected electron, the wave functions obtained with the oxygen core (Section 2.1) were judged to be sufficiently accurate (*cf.* I). $\mathscr{C}_p$ was estimated to be 0·96 for nitrogen and 0·98 for carbon. The experimental values of I for these elements are 14·46 e.V. and 11·20 e.V. respectively.§ Using the data indicated, the absorption cross-sections at the spectral heads were evaluated from the dipole moment and dipole velocity formula. The figures obtained are quoted in Table II. As before, the alternative formulae yield appreciably different results of which, it is suggested, the mean be taken. The final averaged cross-sections are about half those given by the crude early work. Such a correction factor is by no means excessive in calculations of this type.

TABLE II

Continuous radiative absorption cross-sections of atomic nitrogen and carbon at spectral head

[A in units 10^{-18} cm.^{2}]

Atom	Value from dipole moment formula	Value from dipole velocity formula	Recommended value
Nitrogen	$10{\cdot}_2$	$7{\cdot}_7$	9
Carbon	$12{\cdot}_5$	$9{\cdot}_0$	11

2.4. *Variation of absorption cross-sections with energy.*—The results so far given refer only to the spectral heads. It is known that the cross-sections of the atoms under discussion vary only slowly with energy ‖ (except of course at an ionization potential). In view of this the labour of evaluating directly the absolute cross-sections throughout the continua did not appear justified at the present stage. Instead the cross-section energy curves were calculated from an approximate general formula that has been derived by Bates ¶, the exchange

* For oxygen the corresponding overestimation is by a factor of 1·7: for neon the factor is probably rather smaller.

† F. R. Brown, J. H. Bartlett and C. G. Dunn, *Phys. Rev.*, **44**, 296, 1933. C. Torrance, *Phys. Rev.*, **46**, 388, 1934.

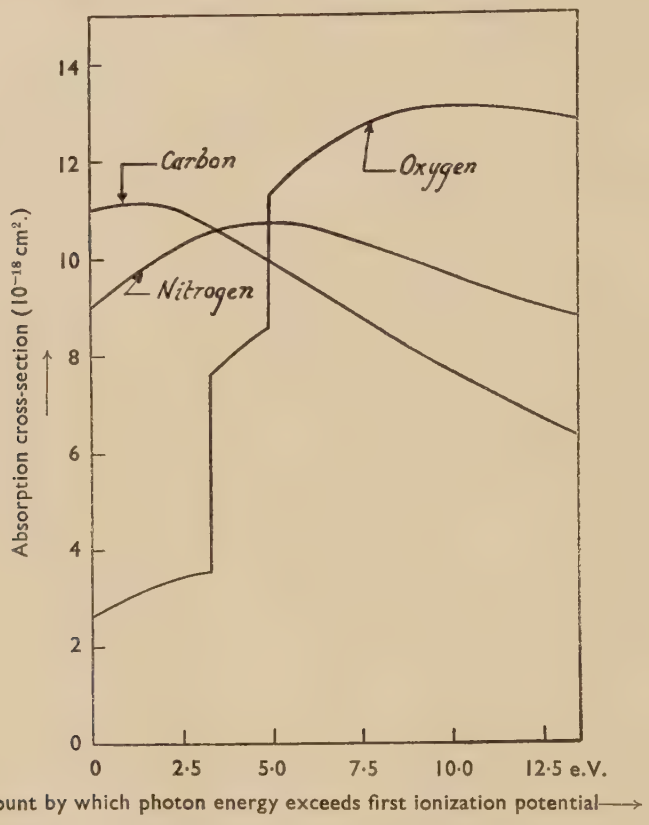
‡ A. Jucys, *Proc. Roy. Soc.* A, **173**, 59, 1939. D. R. Hartree and W. Hartree, *Proc. Roy. Soc.* A, **193**, 299, 1948.

§ J. C. Boyce, *loc. cit.*, 1941.

‖ D. R. Bates, *loc. cit.*, 1939.

¶ D. R. Bates, *M.N.*, **106**, 423, 1946.

wave functions being used to determine the values of the parameters involved in this. Each was then made to pass through the previously computed point at the spectral head (i. e. zero energy) by multiplying the cross-section scale by a suitable constant (which, it is satisfactory to note, was in no instance very different from unity). The diagram shows the curves thus obtained.



Absorption cross-section curve for atomic oxygen, nitrogen and carbon.
(*Note: The breaks in the oxygen curve are at the second and third ionization potentials.*)

Finally, acknowledgements are due to Professor H. S. W. Massey, F.R.S., for helpful discussions, and to the Ministry of Education for the award of a grant to one of us (M. J. S.).

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NOTES ON THE PULSATIONS OF STARS

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Summary

The amplitude of the fourth harmonic mode of pulsation of a model proposed by T. G. Cowling has been obtained by numerical integration. This mode is found to be more stable than the fundamental. A method is given for obtaining approximate formulae for the amplitude of any mode.

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1. T. G. Cowling (1) has studied the problem of the stability of simple stellar models undergoing spherically symmetrical pulsations. With the rate of energy generation per unit mass ϵ depending on the density ρ and the temperature T, varying according to the law $\epsilon = \epsilon_0 \rho T^\beta$, he finds that a certain model is stable, provided $\beta < 448$. This paper contains remarks on such pulsations and an account of calculations performed in connection with the higher harmonics.

2. Some qualitative results can be obtained from general physical arguments. In the equilibrium state of the star, energy is generated (mainly in the central regions) and transmitted through the star to the surface. When the star pulsates, all the physical variables will oscillate about their equilibrium values; in particular, the rate of energy generation and transmission will vary. It is to be expected that if the difference between the variations of these two factors, that is, the resultant variation in energy content, is positive, when suitably averaged over the star, then the energy of the pulsation will increase, and the star will be unstable. Since the amount of energy radiated by the star during one cycle of the pulsation is only a very small fraction of the total energy of the star, the effect of the variation of ϵ and of κ, the opacity coefficient, on the amplitude of the pulsation will be slight, and will only become appreciable after many cycles; by this time the pulsation will have travelled all over the star, therefore any criterion of stability must involve the whole star, and not just isolated regions.

The parts played by the variations in ϵ and κ may be studied by comparing the pulsations of the model with hypothetical pulsations of the same model in which ϵ and κ are supposed not to vary from their equilibrium values. Consider a sphere, concentric with the star, containing a fixed mass of gas; consider also the fundamental mode of pulsation. Even when ϵ and κ do not vary, the pulsations are not completely adiabatic, since the temperature gradient and the area of the sphere, on which the transmission of energy depends, will vary with the pulsation. These two effects will be in opposition, since when the star contracts its temperature increases; it so happens, however, that the resultant effect of the contraction is an increase in the transmission of energy. This will tend to decrease the maximum temperature of the pulsation, that is, to decrease the maximum pressure and thus the restoring force. The amplitude of the pulsation will therefore tend to decrease with time. Now if κ has the form $\kappa = \kappa_0 \rho / T^{\frac{7}{2}}$, adiabatic changes will

cause κ to decrease as the temperature increases, so the variation in κ will also have a stabilizing effect; the variation in ϵ will have the opposite effect, since more energy will be generated as the star contracts, so increasing the maximum temperature and the restoring force. Thus one can expect an upper limit to β in order that the pulsations shall be stable, this limit depending on the model chosen.

3. In Cowling's analysis m, the mass enclosed by a sphere of radius r concentric with the star, is used as independent variable. F is the total flux of radiation through the sphere. Relations between the variables in the equilibrium state are obtained from the energy equation and the equations of motion and continuity, assuming that radiation pressure is negligible; the pulsations are described by allowing first-order variations in these variables, δF in F, etc. By assuming that the pulsations are almost adiabatic, that is, by taking the resultant variation in energy content per unit mass $\delta\epsilon - d(\delta F)/dm$ as a small perturbation, the condition that the amplitude of the pulsations shall decrease with time is found to be that

$$\int_0^M \frac{\delta\rho}{\rho}\left(\delta\epsilon - \frac{d(\delta F)}{dm}\right) dm < 0,$$

M being the total mass of the star. This expression exhibits the qualities that were expected of it. Using the general form $\kappa = \kappa_0 \rho^n / T^S$, this criterion becomes

$$\beta < \frac{1}{\gamma - 1}\left[\tfrac{1}{2}\{S(\gamma-1) - n + 3\gamma - 4\}\left(1 + \frac{1}{\Delta'}\left(\frac{\delta\rho}{\rho}\right)_M^2\right) - 1 + \frac{\sigma^2 R^2}{3GM}\cdot\frac{\Delta}{\Delta'}\right],$$

where γ is the ratio of the specific heats of the gas, G the gravitational constant, σ the frequency of the pulsation, and R the radius of the star. Δ and Δ' are given by

$$\frac{LR^3\Delta}{3M} = -\int_0^M \frac{\delta\rho}{\rho}\frac{d}{dm}\left(\frac{Fr^3\delta r}{m}\right) dm, \qquad L\Delta' = \int_0^M \epsilon\left(\frac{\delta\rho}{\rho}\right)^2 dm,$$

where L is the equilibrium value of the total flux of radiation from the star. For large values of β, Δ' is approximately equal to $(\delta\rho/\rho)_0^2$, since the pulsation is almost constant near the centre.

The limiting value of β is seen to depend directly on the ratio $(\delta\rho/\rho)_M^2 : (\delta\rho/\rho)_0^2$, so that centrally condensed models, for which this ratio is large, will tend to be stable. The term $(\sigma^2 R^3/3GM)(\Delta/\Delta')$ will increase with the frequency, but it is not the most important for the cases Cowling considers.

In the second paper (1) the analysis is extended to consider models whose central regions are in convective equilibrium. The limiting value of β is increased, due to turbulence, in the convective case by a small term, zero for the case $\gamma = \frac{5}{3}$. The properties of a model of this type are evaluated by numerical integration; in the central regions of the model, conditions are adiabatic, with $\gamma = \frac{5}{3}$, while in the outer regions there is radiative equilibrium. An important feature is that the ratio of central to mean density is 37, comparable with the value 54 for the Standard Model (the polytrope $n = 3$). To find the form of the fundamental mode of pulsation the variational equations are put into the form

$$\frac{d}{dx}\left(\frac{\delta r}{r}\right) = -\frac{1}{x}\left(\frac{\delta\rho}{\rho} + 3\frac{\delta r}{r}\right), \qquad \frac{d}{dx}\left(\frac{\delta\rho}{\rho}\right) = \frac{x}{y}\left\{\frac{3}{2}\left(\Sigma^2 + \frac{4w}{x^3}\right)\frac{\delta r}{r} + \frac{5w}{2x^3}\frac{\delta\rho}{\rho}\right\}, \quad (1)$$

where x, y, w and Σ^2 are dimensionless variables proportional to r, T, m and σ^2; these equations are integrated numerically with the boundary conditions that

$\delta r/r$ and $\delta\rho/\rho$ should be finite at the centre and at the boundary. The eigenvalue of Σ^2 is found to lie between 0·086 and 0·0868 and $(\delta\rho/\rho)_M : (\delta\rho/\rho)_0$ is 38, giving the limit for β as 448.

4. In order to see if the fundamental mode is the least stable for Cowling's model the equations (1) were integrated numerically for one of the higher harmonics. The solutions were started off from the origin by means of power series, then carried out to the boundary by means of finite differences in the usual way. Four sets of solutions were obtained corresponding to four values of Σ^2; in this way the eigenvalue was straddled. For the two values of Σ^2 adjacent to the eigenvalue the equations were integrated inwards from the boundary (in the same way as the outward integrations) and by comparing ratios of $\delta r/r$ and $\delta\rho/\rho$ for the inside and outside solutions at a suitably chosen point the eigenvalue of Σ^2 was estimated as 0·391. This corresponded to the fourth harmonic (three nodes). The estimated value of $(\delta\rho/\rho)_M : (\delta\rho/\rho)_0$ was 3140, giving as the limit for β the value 4×10^7. Thus this mode is much more stable than the fundamental.

5. After the above work had been carried out, a transformation was obtained which makes the behaviour of the solutions more transparent. With $Z = 2/x - 5W/4x^2y$, the substitution

$$\frac{\delta r}{r} = \exp\{-\int^x Z\,dx\}\,\phi, \tag{2}$$

gives the equation for ϕ

$$\frac{d^2\phi}{dx^2} + p^2\phi = 0, \tag{3}$$

where

$$p^2 = \frac{3}{2y}\left(\Sigma^2 - \frac{w}{x^3}\right) - Z^2 - \frac{dZ}{dx}.$$

This equation is much easier to integrate numerically than the equations (1), since it is easy to compute the function p^2, whose form for Cowling's model is shown in Fig. 1. To avoid awkward singularities the transformation is written

$$\frac{\delta r}{r} = \frac{X^\alpha}{x^2(X-x)^\alpha}\exp\{-\int_0^x (Z - 2/x + \alpha/(X-x))\,dx\}\,\phi,$$

where $\alpha = 5W/4X^2D$ and X, W and D are the boundary values of x, w and dy/dx.

From the following argument it appears that the ratio $(\delta\rho/\rho)_M : (\delta\rho/\rho)_0$ increases as Σ^2 increases, implying that the higher modes of oscillation are more stable. From equations (1) the boundary conditions on $\delta r/r$ and $\delta\rho/\rho$ give the relations

$$\left(\frac{\delta\rho}{\rho}\right)_0 = -3\left(\frac{\delta r}{r}\right)_0, \qquad \left(\frac{\delta\rho}{\rho}\right)_M = -\frac{3(\Sigma^2 + 4W/X^3)}{5W/X^3},$$

and W/X^3 is small compared with Σ^2. From equation (3), where p^2 is positive, ϕ is oscillatory, and has roughly constant amplitude throughout this region independent of Σ^2; in the end regions, where p^2 is negative, ϕ is convex to the x-axis and goes to zero, at the centre like x^2, and at the boundary like $(X-x)^\alpha$. The variation of $(\delta r/r)_M : (\delta r/r)_0$ with Σ^2 is just the variation in the ratio of the value of $(X-x)^{-\alpha}\phi$ at the boundary to the value of $x^{-2}\phi$ at the centre, and this

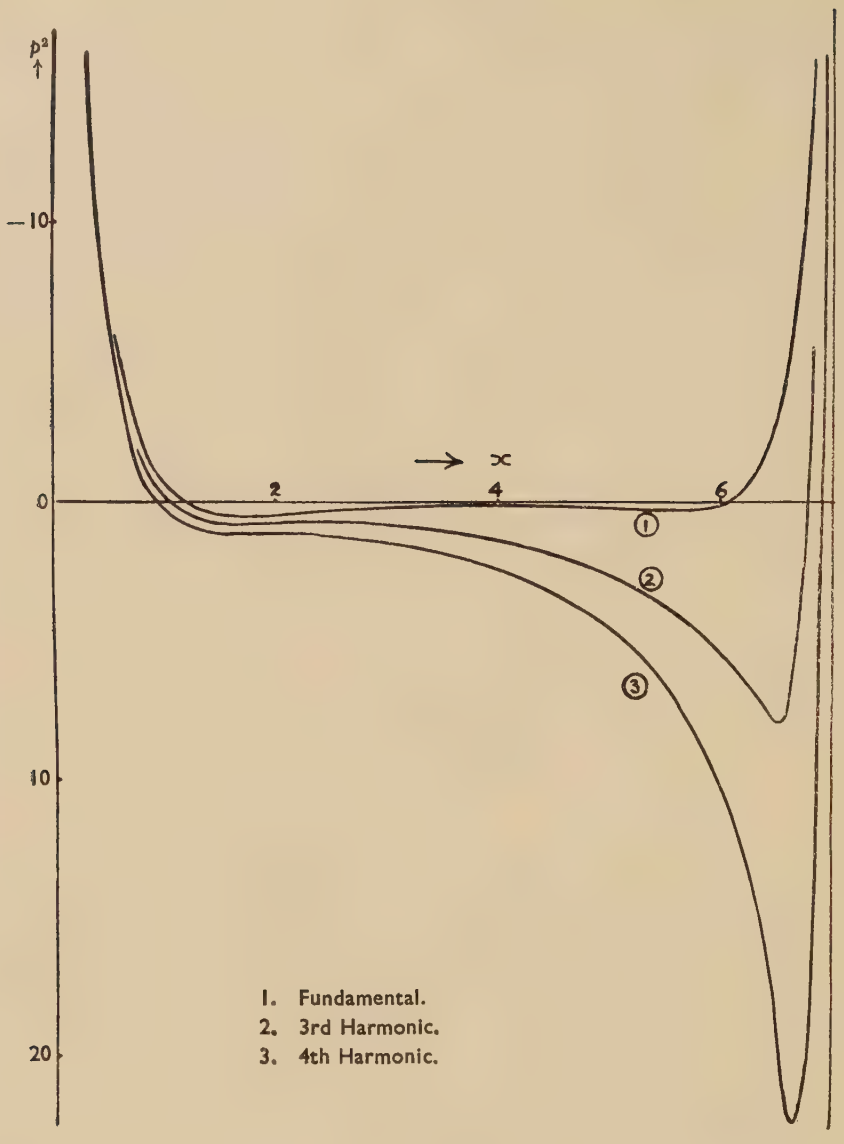

FIG. 1.—*Graph of* p^2.

ratio is approximately the change in $(X-x)^{-\alpha}\phi$ between the boundary and x_2, the outer zero of p^2, times the change in $x^{-2}\phi$ between x_1, the inner zero of p^2, and the centre. By using series expansions of p^2 near the centre and near the boundary, and assuming that the series solutions of equations (3) hold out to x_1 or x_2, it is found that, for large Σ^2,

$$\{(X-x)^{-\alpha}\phi\}_X/\{(X-x)^{-\alpha}\phi\}_{x_2}=1{\cdot}83-0{\cdot}05/\Sigma^2\ldots,$$

$$(x^{-2}\phi)_{x_1}/(x^{-2}\phi)_0=1{\cdot}95-0{\cdot}02/\Sigma^2\ldots.$$

Thus $(\delta r/r)_M:(\delta r/r)_0$ will tend towards a value independent of Σ^2, and $(\delta\rho/\rho)_M:(\delta\rho/\rho)_0$ will increase with Σ^2.

6. The substitution $\phi = \exp\{i\int^x \eta\, dx\}$ gives for η the equation

$$\frac{d\eta}{dx} = i(p^2 - \eta^2).$$

An approximate solution of this is

$$\eta = \pm p + \frac{i}{2p}\frac{dp}{dx},$$

giving

$$\phi = \frac{1}{p^{\frac{1}{2}}}(A\exp\{i\int^x p\, dx\} + B\exp\{-i\int^x p\, dx\}),$$

the Wentzel-Kramers-Brillouin approximation. Although the approximation is not good near the zeros of p^2 (the condition of validity being that $(1/p^2)(dp/dx) \ll 1$),

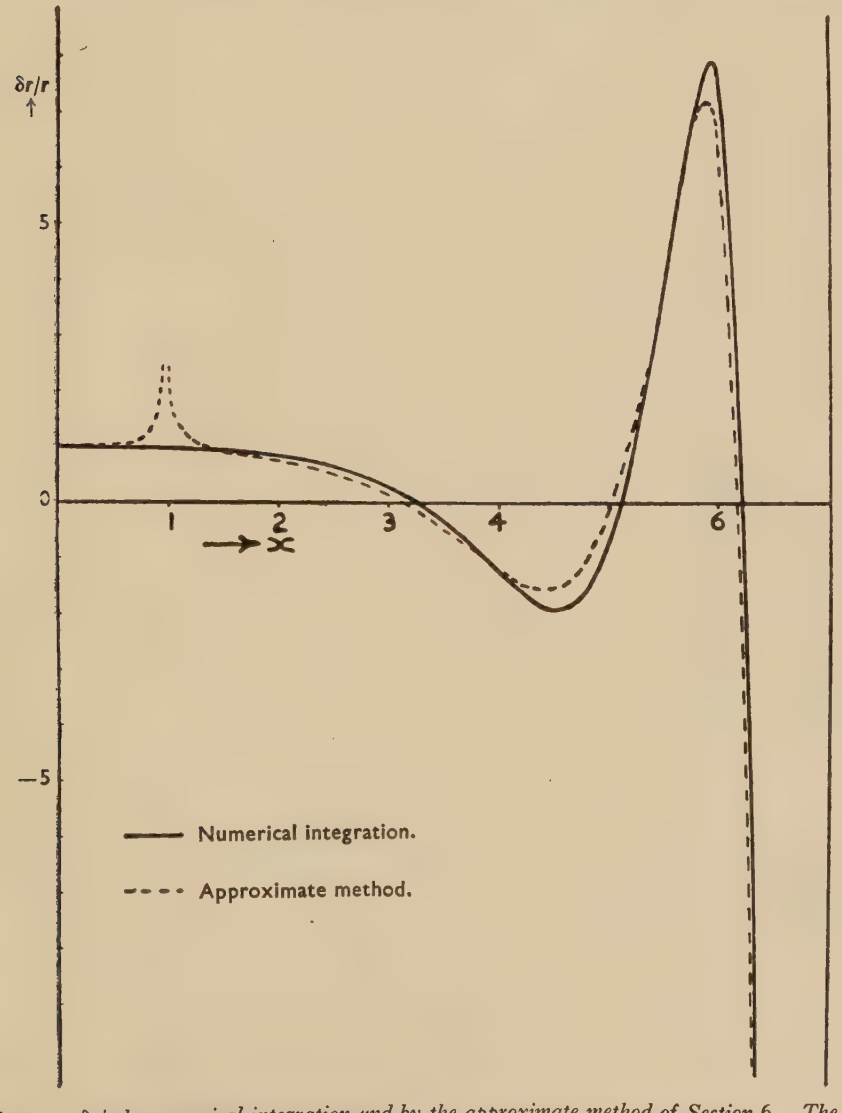

FIG. 2.—*δr/r by numerical integration and by the approximate method of Section 6. The values reached at the boundary are −330 and −270 respectively.*

it can be shown (**2**) that the following are the solutions appropriate to the problem:

$$\phi = \frac{1}{|p|^{\frac{1}{2}}} \exp\left(-\int_x^{x_1} |p|\, dx\right) \quad \text{for} \quad 0 < x < x_1, \tag{4a}$$

$$\phi = \frac{2}{p^{\frac{1}{2}}} \cos\left(\int_{x_1}^{x} p\, dx - \pi/4\right) \quad \text{for} \quad x_1 < x < x_2, \tag{4b}$$

$$\phi = \frac{1}{|p|^{\frac{1}{2}}} \exp\left(-\int_{x_2}^{x} |p|\, dx\right) \quad \text{for} \quad x_2 < x < X. \tag{4c}$$

In order that the solutions (4*b*) and (4*c*) fit at x_2 the condition

$$\int_{x_1}^{x_2} p\, dx = (n + \tfrac{1}{2})\pi \qquad (n = \text{integer}) \tag{5}$$

is required. The approximation also breaks down near the origin and near the boundary, but in these regions the power-series solutions may be used; these may be fitted to (4*a*) and (4*c*) to give an approximate solution covering the whole range. The ratio $(x^{-2}\phi)_0 : \{(X-x)^{-\alpha}\phi\}$ can thus be obtained in a closed form, Σ^2 being determined from the condition (5).

The function $\delta r/r$ was calculated by this method for the fourth harmonic and is shown in Fig. 2, together with the eigenfunction given by numerical integration. (This was obtained by interpolation from the two functions corresponding to the values of Σ^2 adjacent to the eigenvalue.) For the value of Σ^2 obtained in Section 4 the integral in condition (5) was $3{\cdot}63\pi$. The value of this integral for the fundamental mode was in error by about 80 per cent; this is to be expected, however, since the method is accurate only for the higher harmonics.

7. The high stability of Cowling's model seems to be due to the fact that when it pulsates the amplitude of the pulsation is much larger in the outer regions than at the centre. This is connected with the fact that the model has its mass concentrated towards the centre. Indeed, according to Schwarzschild (**3**) the standard model has an even larger value of $(\delta\rho/\rho)_M : (\delta\rho/\rho)_0$ and hence is even more stable than Cowling's model.

The author wishes to thank Professor R. E. Peierls for his invaluable help throughout the work, and Dr M. Krook for many helpful discussions. His thanks are also due to the Department of Scientific and Industrial Research for the grant under which this work was done.

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(1) T. G. Cowling, *M.N.*, **94**, 768, 1934; *M.N.*, **96**, 42, 1935.
(2) See, for example, E. C. Kemble, *Fundamental Principles of Quantum Mechanics*, New York, p. 90, 1937.
(3) M. Schwarzschild, *Ap. J.*, **94**, 245, 1941.

ANHARMONIC PULSATIONS OF THE STANDARD MODEL: COMMENSURABLE PERIODS

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Summary

The equations of anharmonic pulsation are obtained for $\alpha = 0{\cdot}4$ and solved by the use of Fourier series. The skewness of the velocity curve obtained is in conformity with observation. A method is developed for obtaining solutions adjacent to the periodic solution, and the bearing of these upon observation is discussed.

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1. *Introduction.*—The interaction of two modes in the theory of Cepheid pulsation was first studied by Woltjer.* He started with equations which were virtually of the same form as those developed later by Rosseland in the theory of anharmonic pulsations †, but he transformed them to a set of new variables commonly used in celestial mechanics. Then taking the case in which the periods of two modes are nearly commensurable, and limiting himself only to the dominant term in the Hamiltonian, he was able to solve the equations explicitly and obtain a periodic solution. He was also able to obtain solutions adjacent to the strictly periodic solution by the method of variation of parameters. Numerical computation for the standard model using these methods were made by Miss Kluyver for $\gamma = 20/13$.‡

The periodic solution given by Woltjer suffers from a twofold defect. Firstly, it gives only a first-order approximation to the solution, and secondly it neglects the other terms in the Hamiltonian. For these reasons the actual solution may differ considerably from Woltjer's solution. In a previous paper§ the author has shown how the equations of anharmonic pulsation can be solved by the use of Fourier series. With a slight modification the method can also be used in the case of approximate commensurability of the periods, that is, when the period of a higher mode is nearly half the period of the fundamental mode. This method is free from the defects of Woltjer's solution; moreover it shows the relationship of Woltjer's solution to the complete solution.

In the present paper the equations of anharmonic pulsation for the standard model are obtained for the value $\gamma = 20/13$, and a solution corresponding to the observed values of the amplitude is given. As is well known ||, the third mode has a period nearly half of the fundamental for this value of γ. Thus there may be a tendency for the free oscillation to be set up in the higher mode.¶ The star, in trying to set up a mean between the two frequencies, may start oscillating in

* J. Woltjer, *M.N.*, **95**, 260, 1935; *B.A.N.*, **8**, 193, 1937.
† S. Rosseland, *M.N.*, **103**, 239, 1943.
‡ H. A. Kluyver, *B.A.N.*, **7**, 313, 1936.
§ *M.N.*, **109**, 528, 1949, referred as Paper I.
|| M. Schwarzschild, *Ap. J.*, **94**, 245, 1941.
¶ J. Woltjer, *B.A.N.*, **8**, 193, 1937.

a manner which does not correspond to the truly periodic solution but only to an adjacent solution. A method is developed for obtaining the adjacent solutions when all the terms are taken into account. It is found that, as in Woltjer's case, they show a second period considerably larger than the regular period of oscillation.

2. *Equations of the problem.*—We start in a way slightly different from the previous paper. The equations of anharmonic pulsation are given by *

$$I_k \ddot{q}_k = -\frac{\partial H}{\partial q_k}, \quad (k = 1, 2, \ldots) \tag{1}$$

where the displacement

$$\frac{r - r_0}{r_0} = \sum_k \eta_k(r_0) q_k(t), \tag{2}$$

$$I_k = \int_0^R \eta_k^2 r_0^4 \rho_0 \, dr_0, \tag{3}$$

and

$$H = \int_0^R \left(\tfrac{1}{2}\dot{r}^2 + \frac{1}{\gamma - 1}\frac{P}{\rho} - g_0 \frac{r_0^2}{r} \right) r_0^2 \rho_0 \, dr_0, \tag{4}$$

r, ρ, P denoting the distance from the centre, density and pressure, and the suffix zero denoting their equilibrium values. The functions η_k are the amplitudes of the various modes for infinitesimal oscillations.

By using the equation of continuity and the relation (2), H can be expanded in powers of q_k. The linear terms are absent, the second-degree terms are

$$\sum_k \tfrac{1}{2}\sigma_k^2 I_k q_k^2, \tag{5}$$

where $2\pi/\sigma_k$ is the period of oscillation of the kth mode for infinitesimal amplitude; and the third-degree terms are

$$-\sum_{ijk} 2 f \gamma D_{ijk} q_i q_j q_k, \tag{6}$$

where D_{ijk} has a value defined in the previous paper † and f is a numerical factor taking the value 1 for terms of the type $q_i q_j q_k$, $\frac{1}{2}$ for terms of the type $q_j q_k^2$ and $\frac{1}{6}$ for terms of the type q_k^3. Terms of the fourth and higher degrees are neglected in the present discussion.

With the exception of the following section we shall assume everywhere that the functions η_k are normalized to unity at the star's surface; and we shall use the time variable $\tau = \sigma_1 t$.

3. *Woltjer's particular solution.*—We shall first show that a solution similar to Woltjer's may be obtained even without the change of variables. Let η_1 and η_2 be the modes with nearly commensurable periods, and following Woltjer let us suppose that the only third-degree term of importance in the Hamiltonian is the term in $q_1^2 q_2$. Thus the equations for the commensurable modes may be written as

$$\ddot{q}_1 + \sigma_1^2 q_1 = 2D q_1 q_2, \tag{7.1}$$

$$\ddot{q}_2 + \sigma_2^2 q_2 = D q_1^2, \tag{7.2}$$

where $D = \gamma D_{112}$ and η_1, η_2 are normalized in a manner such that $I_1 = I_2 = 1$.

* S. Rosseland, *M.N.*, **103**, 240, 1943; *The Pulsation Theory of Variable Stars*, Oxford, p. 102, 1949.

† Paper I, equation (8).

Assume a solution of the form

$$q_1 = a_1 \cos \sigma t, \qquad q_2 = b_2 \cos 2\sigma t. \tag{8}$$

Substitution in (7) gives

$$a_1(\sigma_1^2 - \sigma^2) \cos \sigma t = D a_1 b_2 (\cos \sigma t + \cos 3\sigma t), \tag{9.1}$$

$$b_2(\sigma_2^2 - 4\sigma^2) \cos 2\sigma t = \tfrac{1}{2} D a_1^2 (1 + \cos 2\sigma t). \tag{9.2}$$

Equating the coefficients of $\cos \sigma t$ and $\cos 2\sigma t$, we obtain

$$a_1(\sigma_1^2 - \sigma^2) = D a_1 b_2, \qquad b_2(\sigma_2^2 - 4\sigma^2) = \tfrac{1}{2} D a_1^2. \tag{10}$$

We may ignore the factor $a_1 = 0$, which corresponds to the excitation of the second mode with the complete absence of the first. The other factor along with the second equation gives, on the elimination of σ,

$$b_2 = \frac{\sigma_1^2 - \frac{1}{4}\sigma_2^2}{2D} - \sqrt{\left\{\left(\frac{\sigma_1^2 - \frac{1}{4}\sigma_2^2}{2D}\right)^2 + \tfrac{1}{8} a_1^2\right\}}, \tag{11}$$

taking that value of b_2 which vanishes with a_1.

The derivation of the equations (10) from (9) virtually amount to neglecting the term in $\cos 3\sigma t$ and the term free from t in (9). It will be seen in the next section that they can be balanced by adding higher-degree terms to q_1 and q_2. Their neglect is equivalent to the neglect of the terms $\cos(2w_1 + w_2)$ and $\cos w_2$ in the Hamiltonian by Woltjer.* To a first order, the expression (11) is equivalent to Woltjer's solution. The equivalence may be shown by putting

$$D = K\sigma_1 \sqrt{(2\sigma_2)}, \qquad a_1 = \sqrt{(2J_1/\sigma_1)}, \qquad b_2 = -\sqrt{(2J_2/\sigma_2)},$$

K, J_1, J_2 being the constants used by Woltjer; and then taking the first-order approximation $(\sigma_1 + \frac{1}{2}\sigma_2)/2\sigma_1 = 1$.

For comparison we give below numerical solutions for the standard model with $\gamma = 20/13$. In this case the first and third modes are nearly commensurable; and a solution according to (11), but reduced to the normalization $\eta_1 = \eta_3 = 1$ at the star's surface, is

$$q_1 = 0{\cdot}05894 \cos \sigma t, \qquad q_3 = -0{\cdot}01698 \cos 2\sigma t. \tag{12.1}$$

The corresponding Woltjer solution is

$$q_1 = 0{\cdot}05894 \cos \sigma t, \qquad q_3 = -0{\cdot}01724 \cos 2\sigma t. \tag{12.2}$$

In the above equations σ is very nearly equal to σ_1, being given by

$$\sigma^2 = 1{\cdot}00016\sigma_1^2. \tag{13}$$

4. *Complete solution.*—The complete solution of equations (7) can be obtained by taking instead of (8) the relations

$$q_1 = a_0 + a_1 \cos \sigma t + a_2 \cos 2\sigma t + a_3 \cos 3\sigma t + \ldots, \tag{14.1}$$

$$q_2 = b_0 + b_1 \cos \sigma t + b_2 \cos 2\sigma t + b_3 \cos 3\sigma t + \ldots, \tag{14.2}$$

substituting in (7), and equating to zero the various coefficients of $\cos k\sigma t$. But instead of solving the limited equations (7) we shall take the equations with all the quadratic terms on the right-hand side. These may be written as

$$\frac{d^2 q_1}{d\tau^2} + q_1 = A_1 q_1^2 + 2B_1 q_1 q_2 + C_1 q_2^2, \tag{15.1}$$

$$\frac{d^2 q_2}{d\tau^2} + \beta_2 q_2 = A_2 q_1^2 + 2B_2 q_1 q_2 + C_2 q_2^2, \tag{15.2}$$

* S. Rosseland, *The Pulsation Theory of Variable Stars*, Oxford, p. 115, 1949.

where $\tau=\sigma_1 t$, $\beta_2=\sigma_2^2/\sigma_1^2$, $A_1=\gamma D_{111}/\sigma_1^2 I_1$, $A_2=\gamma D_{112}/\sigma_1^2 I_2$, ...

Putting $n\tau$ for σt in (14) and substituting in (15), we find that the coefficients a_0, a_1, ... are connected by relations of the type *

$$(k^2n^2-1)a_k+A_1[\tfrac{1}{2}\sum_{i=0}^{k} a_i a_{k-i}+\sum_{i=0}^{\infty} a_i a_{k+i}]$$
$$+B_1[\sum_{i=0}^{k} a_i b_{k-i}+\sum_{i=0}^{\infty}(a_i b_{k+i}+b_i a_{k+i})]$$
$$+C_1[\tfrac{1}{2}\sum_{i=0}^{k} b_i b_{k-i}+\sum_{i=0}^{\infty} b_i b_{k+i}]=0 \qquad (16)$$

($k=1, 2, \ldots$), and similar relations with $(k^2n^2-\beta_2)b_k$ replacing the first term and A_2, B_2, C_2 replacing A_1, B_1, C_1. For $k=0$ the relations are slightly different, the second term in each square bracket being multiplied by $\frac{1}{2}$.

Let us write down in full the two equations corresponding to the equations (10). They are

$$(n^2-1)a_1+A_1[2a_0a_1+a_1a_2+a_2a_3+\ldots]+2B_1[a_1b_0+a_0b_1+\tfrac{1}{2}(a_1b_2+a_2b_1)+\ldots]$$
$$+C_1[2b_0b_1+b_1b_2+\ldots]=0, \qquad (17)$$

$$(4n^2-\beta_2)b_2+A_2[\tfrac{1}{2}a_1^2+2a_0a_2+a_1a_3+\ldots]+2B_2[\tfrac{1}{2}a_1b_1+a_0b_2+a_2b_0+\ldots]$$
$$+C_2[\tfrac{1}{2}b_1^2+2b_0b_2+\ldots]=0. \qquad (18)$$

In case of approximate commensurability of the two periods β_2 is nearly 4, so that $4n^2-\beta_2$ is small and consequently b_2 is large. Therefore we assume a_1 and b_2 both to be small quantities of the first order, and to obtain a relation between them we eliminate n^2 between the two equations above. As a first approximation we neglect all the quadratic terms except the second-order terms $B_1a_1b_2$ in the first equation and $\frac{1}{2}A_2a_1^2$ in the second. This gives the solution obtained in the previous section. Using this approximation we next calculate a_0, a_2, ..., b_0, b_1, ... in terms of a_1 from the other equations, and use them to find a second approximation to the relation between b_2 and a_1; and so on.

It may be noted that in the case of commensurable periods n^2-1 becomes a small quantity of the first order, and the order of the quantities a_3, a_4, b_1, b_3 and b_4 is lowered to two. But actually in the case treated below the lowering of the order of n^2-1, a_3 and a_4 is not apparent, as the coefficient B_1 is very small.

5. *Application to the standard model.*—For the standard model the second mode has a period which is greater than half the fundamental period for admissible values of the effective ratio of specific heats. But the period of the third mode varies from a value greater than half to a value less than half of the fundamental period. For a value of γ slightly less than the value given by $\alpha=3-4/\gamma=0{\cdot}4$, we get exact commensurability. The amplitudes of the various modes of oscillation for $\alpha=0{\cdot}4$ have been given by Schwarzschild.† Using these we shall compute the coefficients in the equations of anharmonic pulsation and solve them.

For the purpose of computation it is more convenient to use the Emden variables ξ, θ for the polytrope $n=3$, and the corresponding eigenvalue ω_1, instead of r_0, ρ_0, P_0 and σ_1. In terms of these variables the coefficients $\gamma D_{ijk}/\sigma_1^2 I_k$

* Paper I, equation (15).
† *Loc. cit.*

may be written $D^0_{ijk}/\omega_1^2 I^0_k$, where

$$D^0_{ijk} = -2\left(3-\frac{4}{\gamma}\right)(3\gamma+1)\int_0^{\xi_1}\theta^3\theta'\xi^3\eta_i\eta_j\eta_k\,d\xi + \tfrac{1}{2}(3\gamma-1)\int_0^{\xi_1}\theta^4\xi^4(\eta_i\eta_j'\eta_k' + \eta_i'\eta_j\eta_k' + \eta_i'\eta_j'\eta_k)\,d\xi + \tfrac{1}{2}(\gamma+1)\int_0^{\xi_1}\theta^4\xi^5\eta_i'\eta_j'\eta_k'\,d\xi, \quad (19.1)$$

$$I^0_k = \int_0^{\xi_1}\theta^3\xi^4\eta_k^2\,d\xi \quad (19.2)$$

and

$$\omega_1^2 = \frac{\rho_c R^2\sigma_1^2}{\xi_1^2 P_c\gamma} = \frac{\sigma_1^2}{\pi\gamma G\rho_c}, \quad (19.3)$$

the dashes representing differentiation with respect to ξ.

The coefficients are calculated by numerical integration, taking the amplitudes with the normalization $\eta_k = 1$ at the surface of the star. The equations for the first and third modes are found to be

$$\ddot{q}_1 + q_1 = 0{\cdot}9087q_1^2 + 0{\cdot}0190q_1q_3 + 0{\cdot}0409q_3^2, \quad (20.1)$$

$$\ddot{q}_3 + 3{\cdot}772q_3 = 2{\cdot}235q_1^2 + 19{\cdot}23q_1q_3 + 7{\cdot}976q_3^2, \quad (20.2)$$

where $\ddot{q}_k \equiv d^2q_k/d\tau^2$.

It may be noticed that the coefficients of q_1q_3 and q_3^2 in the first equation are very small; also the commensurability of the periods is not very near. For these reasons it is much more convenient in practice to find out $a_0, a_2, \ldots$ in terms of a_1, neglecting the terms in q_3 at first. Using these values the equations for the b's are then solved as simultaneous equations. The correction for the effect of q_3 upon q_1 can be applied at a next approximation. A solution chosen so that $q_1(0) = 0{\cdot}06$ is

$$q_1 = {\cdot}001583 + {\cdot}05894\cos n\tau - {\cdot}000527\cos 2n\tau + {\cdot}000004\cos 3n\tau, \quad (21.1)$$

$$q_3 = {\cdot}00091 - {\cdot}00218\cos n\tau - {\cdot}01302\cos 2n\tau + {\cdot}00139\cos 3n\tau - {\cdot}00012\cos 4n\tau, \quad (21.2)$$

where

$$n^2 = 0{\cdot}9977.$$

It is seen that the increase in period which is very slight (0·12 per cent) is even less than the increase in the non-commensurable case (0·23 per cent). The reason is that the effect of the commensurability is to decrease the period [*cf.* equation (13)]. A comparison with the first-order solution (12) shows that besides the addition of extra terms, the coefficient of $\cos 2n\tau$ in q_3 has decreased considerably.

The displacement q_b at the boundary due to these modes is $q_1 + q_3$, i. e.

$$q_b = {\cdot}00249 + {\cdot}05676\cos n\tau - {\cdot}01355\cos 2n\tau + {\cdot}00139\cos 3n\tau - {\cdot}00012\cos 4n\tau. \quad (22)$$

The ratio K of the time of rise of the velocity from minimum to maximum to the total period is 0·306 for this curve.

6. *Solution including the second mode.*—In the solution found above, the second mode was neglected; but actually the second mode, being the first overtone, would be quite effective in modifying the oscillations. We consider below the solution including the second mode.

As q_3 is large compared with q_1, it would not be advisable to neglect quadratic terms involving q_3, in the equations. So we have to calculate the coefficients

of all the quadratic terms. The equations for $\alpha = 0{\cdot}4$ are found to be

$$\ddot{q}_1 + q_1 = {\cdot}9087q_1^2 + {\cdot}3819q_1q_2 + {\cdot}0190q_1q_3 + {\cdot}0409q_3^2 + {\cdot}0788q_2q_3 + {\cdot}1599q_2^2, \tag{23.1}$$

$$\ddot{q}_2 + 2{\cdot}117q_2 = 5{\cdot}140q_1^2 + 8{\cdot}610q_1q_2 + 2{\cdot}121q_1q_3 + 1{\cdot}134q_3^2 + 2{\cdot}890q_2q_3 + 3{\cdot}245q_2^2, \tag{23.2}$$

$$\ddot{q}_3 + 3{\cdot}772q_3 = 2{\cdot}235q_1^2 + 18{\cdot}54q_1q_2 + 19{\cdot}23q_1q_3 + 7{\cdot}976q_3^2 + 19{\cdot}82q_2q_3 + 12{\cdot}63q_2^2. \tag{23.3}$$

The equations though lengthy are not unmanageable; only more approximations are needed to reach the same degree of accuracy. For our solution we have taken the same value of a_1 as in the preceding section. The values of q_1 and q_3 found there were used to find q_2 to a first approximation, and then these were used for further approximations.

The solution is found to be

$$q_1 = {\cdot}00161 + {\cdot}05894\cos n\tau - {\cdot}00053\cos 2n\tau + {\cdot}00001\cos 3n\tau, \tag{24.1}$$

$$q_2 = {\cdot}00452 + {\cdot}00118\cos n\tau - {\cdot}00474\cos 2n\tau + {\cdot}00031\cos 3n\tau - {\cdot}00003\cos 4n\tau, \tag{24.2}$$

$$q_3 = {\cdot}00159 - {\cdot}00095\cos n\tau - {\cdot}01200\cos 2n\tau + {\cdot}00180\cos 3n\tau - {\cdot}00021\cos 4n\tau, \tag{24.3}$$

where

$$n^2 = 0{\cdot}99683.$$

We have neglected the term in $\cos 5n\tau$ which is about a tenth of the term in $\cos 4n\tau$. It may be noted that the effect of q_2, apart from its own contribution, is to diminish the second and third terms in q_3, which amounts to diminishing the skewness due to q_3.

The displacement q_b at the boundary is given by

$$q_b = {\cdot}00772 + {\cdot}05917\cos n\tau - {\cdot}01727\cos 2n\tau + {\cdot}00212\cos 3n\tau - {\cdot}00024\cos 4n\tau. \tag{25}$$

The ratio K for this curve is 0·287.

It is interesting to compare the solution found above with the solution for $\gamma = \frac{5}{3}$. Though for the present case the commensurability of periods is much better than in the case $\gamma = \frac{5}{3}$, the amplitude of q_3 has not increased by a corresponding amount. $4 - \beta_3$, which may be taken as a measure of commensurability, has decreased from a value 0·921 to the value 0·228, a ratio of $\frac{1}{4}$ nearly. But the coefficient c_2 of $\cos 2n\tau$ in q_3, instead of showing a fourfold increase, has increased only 1·17 times. This small increase is due firstly to a decrease of the coefficient of q_1^2 in the equation (23.3) to about half the value for $\gamma = \frac{5}{3}$; and secondly to the moderating influence of the higher-order terms. Strangely enough, the change in other terms is much more than in c_2, which is directly affected by the factor $4 - \beta_3$ [see equation (18)].

7. *Solutions adjacent to the periodic solution.*—Making several assumptions to simplify his analysis, Woltjer has shown that in some cases of commensurable periods a free oscillation in the higher mode may also be excited.* We shall not go into the causes of such an oscillation, but shall only try to find out more

* *Loc. cit.*

accurately the manner of oscillation in the presence of a small free oscillation in the higher mode.

When we integrate the equations (15) to find the periodic solutions $q_1(\tau)$ and $q_2(\tau)$, we can start with any initial value $q_1(0)$, $\dot{q}_1(0)$. The initial values $q_2(0)$, $\dot{q}_2(0)$ are then uniquely determined from the condition that $q_1, \dot{q}_1, q_2, \dot{q}_2$ repeat themselves after some period τ_P.* The solution q_2 found thus, is the forced oscillation in the second mode. If we start with different initial values of q_2 and $\dot{q}_2$, we obtain a solution which may be considered as a superposition of the free oscillation in the second mode over the forced one. This solution will not be periodic, but the deviation from the periodic solution will be small if the initial values are not much different.

We obtain below this deviation from the periodic solution, and the only assumption made is to suppose the deviation so small that its square can be neglected. We put in the equations (15) $q_1+\Delta q_1$ instead of q_1 and $q_2+\Delta q_2$ instead of q_2, where Δq_1 and Δq_2 denote the deviation from the periodic solution q_1 and q_2 found before. Then the differential equations become

$$\Delta\ddot{q}_1+\Delta q_1=2(A_1q_1+B_1q_2)\Delta q_1+2(B_1q_1+C_1q_2)\Delta q_2, \tag{26.1}$$

$$\Delta\ddot{q}_2+\beta_2\Delta q_2=2(A_2q_1+B_2q_2)\Delta q_1+2(B_2q_1+C_2q_2)\Delta q_2, \tag{26.2}$$

where $\Delta\ddot{q}_k$ stands for $d^2\Delta q_k/d\tau^2$.

Writing down the values of q_1 and q_2, these may be arranged as

$$\Delta\ddot{q}_1+(A_0'+A_1'\cos n\tau+A_2'\cos 2n\tau+\ldots)\Delta q_1 +(B_0'+B_1'\cos n\tau+B_2'\cos 2n\tau+\ldots)\Delta q_2=0, \tag{27.1}$$

$$\Delta\ddot{q}_2+(A_0''+A_1''\cos n\tau+A_2''\cos 2n\tau+\ldots)\Delta q_1 +(B_0''+B_1''\cos n\tau+B_2''\cos 2n\tau+\ldots)\Delta q_2=0, \tag{27.2}$$

where A_0', A_1', . . . are known constants.

The method of solution follows the treatment by G. W. Hill of the equations of motion of the lunar perigee.† Assume a solution of the form

$$\Delta q_1=\sum_{k=-\infty}^{\infty}a_k'\cos(n'+kn)\tau, \qquad \Delta q_2=\sum_{k=-\infty}^{\infty}b_k'\cos(n'+kn)\tau. \tag{28}$$

Substitute in (27) and after expressing the products of cosines as sums of cosines, equate the coefficient of $\cos(n'+kn)\tau$ to zero. We get

$$\ldots+\tfrac{1}{2}A_2'a_{k-2}'+\tfrac{1}{2}A_1'a_{k-1}'+\{A_0'-(n'+kn)^2\}a_k'+\tfrac{1}{2}A_1'a_{k+1}'+\tfrac{1}{2}A_2'a_{k+2}'+\ldots +\tfrac{1}{2}B_2'b_{k-2}'+\tfrac{1}{2}B_1'b_{k-1}'+B_0'b_k'+\tfrac{1}{2}B_1'b_{k+1}'+\tfrac{1}{2}B_2'b_{k+2}'+\ldots=0 \tag{29.1}$$

from the first equation, and

$$\ldots+\tfrac{1}{2}A_2''a_{k-2}'+\tfrac{1}{2}A_1''a_{k-1}'+A_0''a_k'+\tfrac{1}{2}A_1''a_{k+1}'+\tfrac{1}{2}A_2''a_{k+2}'+\ldots+\tfrac{1}{2}B_2''b_{k-2}' +\tfrac{1}{2}B_1''b_{k-1}'+\{B_0''-(n'+kn)^2\}b_k'+\tfrac{1}{2}B_1''b_{k+1}'+\tfrac{1}{2}B_2''b_{k+2}'+\ldots=0 \tag{29.2}$$

from the second equation. In the above k can take any integral value, positive or negative. So we get a set of linear equations giving the constants a_k', b_k' in terms of each other. Eliminating the constants a_k', b_k' from these relations we obtain a determinant which gives the values of n'.

It is quite easy to solve the equations by successive approximation, as all the coefficients except A_0' and B_0'' are small and decrease rapidly as we proceed farther from the central term. In the determinant for n', only the diagonal

* M. Schwarzschild and M. P. Savedoff, *Ap. J.*, **109**, 208, 1949.

† G. W. Hill, *Coll. Works*, **1**, 255, 1905 (Carnegie Institute of Washington).

terms are of importance. Therefore as a first approximation we start with either

$$A_0' - n'^2 = 0, \qquad a_0' \neq 0, \tag{30}$$

or

$$B_0'' - n'^2 = 0, \qquad b_0' \neq 0. \tag{31}$$

These respectively correspond to deviation by increasing the primary, and deviation by exciting the secondary. The former is of no value to us as it will lead to just another solution of the original equations (15) with a different value of a_1. The latter gives the solution of the case in which the star has a small free oscillation in the higher mode as well.

Treating the equations (29) as simultaneous equations, other coefficients may be expressed in terms of b_0'. Then using these, an approximation for n' better than (31) may be obtained. The process may be repeated if necessary. Because of the symmetry of the relations (29), the same solution will be obtained if we start with

$$B_0'' - (n' + kn)^2 = 0, \qquad b_k' \neq 0,$$

instead of (31).

We shall solve the case numerically for two modes only, taking the solution (21), and shall find solutions adjacent to it. The equation (27) in this case becomes, writing only the significant terms,

$$\Delta\ddot{q}_1 + (\cdot 9971 - \cdot 1071 \cos n\tau + \cdot 0012 \cos 2n\tau)\Delta q_1 + (-\cdot 0001 - \cdot 0009 \cos n\tau + \cdot 0011 \cos 2n\tau - \cdot 0001 \cos 3n\tau)\Delta q_3 = 0, \tag{32.1}$$

$$\Delta\ddot{q}_3 + (3\cdot 727 - 1\cdot 099 \cos n\tau + \cdot 2178 \cos 2n\tau - \cdot 0223 \cos 3n\tau + \cdot 0019 \cos 4n\tau)\Delta q_3 + (-\cdot 0246 - \cdot 2215 \cos n\tau + \cdot 2527 \cos 2n\tau - \cdot 0267 \cos 3n\tau + \cdot 0023 \cos 4n\tau)\Delta q_1 = 0, \tag{32.2}$$

where $n^2 = 0\cdot 9977$.

The solution of these equations, putting $b_0' = 1$ for convenience, is

$$\Delta q_1 = \cdot 0003 \cos(n' - 3n)\tau - \cdot 0005 \cos(n' - 2n)\tau + \cdot 0034 \cos(n' - n)\tau, \tag{33.1}$$

$$\Delta q_3 = \cdot 001 \cos(n' - 4n)\tau - \cdot 004 \cos(n' - 3n)\tau - \cdot 001 \cos(n' - 2n)\tau + \cdot 195 \cos(n' - n)\tau + \cos n'\tau - \cdot 112 \cos(n' + n)\tau + \cdot 014 \cos(n' + 2n)\tau - \cdot 002 \cos(n' + 3n)\tau, \tag{33.2}$$

where $n' = 1\cdot 919$ and $n = 0\cdot 999$. It is evident that Δq_1 is negligible, and in Δq_3 we may restrict ourselves to three terms only, namely

$$\Delta q_3 = b_0'\{\tfrac{1}{5}\cos(n' - n)\tau + \cos n'\tau - \tfrac{1}{10}\cos(n' + n)\tau\}. \tag{34}$$

Each of these angles can be expressed as the sum of $(n' - 2n)\tau$ and a multiple of $n\tau$. Therefore $q_3 + \Delta q_3$ will show a periodicity of $2\pi/(2n - n')$ besides the original period $2\pi/n$. The ratio of these two periods is

$$\frac{n}{2n - n'} = 12\cdot 6 \tag{35}$$

for the present case. Woltjer's method gives the ratio as 17·2. It should be noted that n'/n is appreciably different from σ_3/σ_1; in fact

$$\frac{\sigma_1}{2\sigma_1 - \sigma_3} = 17\cdot 3. \tag{36}$$

This means that in the presence of the fundamental oscillation, the free oscillation in the third mode has a period different from the period of a star oscillating wholly in the third mode. Equation (36) also shows that Woltjer's method does not give a good approximation for the ratio of the two periods. For actual stars * this ratio is about 70, which leads to the inference that the commensurability for them must be much more exact.

8. *Conclusions.*—For stars with masses varying from 5 to 25 times the mass of the Sun, the ratio of radiation pressure to the total pressure varies from 0·05 to 0·35, assuming the standard model for the star and taking $\mu = 1$. The value of α calculated from the effective ratio of specific heats would vary from 0·50 to 0·26. For δ Cephei, α would be 0·39. Even if the stars are not built on the standard model, the variation in radiation pressure would not be much beyond the above range, and it would be reasonable to suppose α to be in the neighbourhood of 0·4.

The results obtained in the paper show that the standard model for $\alpha = 0{\cdot}4$ is not unsuitable for representing the Cepheid variables from the viewpoint of the skewness of the velocity curve. Contrary to our expectations †, the nearness of the commensurability has not made a large difference, and we are tempted to draw the conclusion that even for smaller α the results would be in conformity with observation.

In comparing the results with observation we should take note of our limitations. The ratio K for the curve (25) is 0·287, whereas the observed values for δ Cephei and η Aquilae are 0·20 and 0·24 respectively.‡ A part of the difference at least is due to the neglect of cubic and higher terms in equations (23). For a curve in which only the terms in $\cos n\tau$ and $\cos 2n\tau$ predominate, K cannot be less than 0·25.

The results of the last section would account well for RR Lyrae type stars. The long-period variation in these stars is due to the interference between the second-order effects of the fundamental oscillation and a small free oscillation in a higher mode, which is nearly commensurable with the fundamental. A comparison of numerical results with the observed values indicates that the commensurability is much closer for the actual stars than in the standard model with $\alpha = 0{\cdot}4$.

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Queen's College,
Oxford:
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* S. Rosseland, *The Pulsation Theory of Variable Stars*, Oxford, p. 109, 1949.
† Paper I, p. 534.
‡ Schwarzschild and Savedoff, *loc. cit.*

AN EARTH MODEL BASED ON A COMPRESSIBILITY-PRESSURE HYPOTHESIS *

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Summary

An Earth model, based on the compressibility-pressure hypothesis earlier put forward by the writer, is constructed. In this model there is a sharp density increase from 3·36 to 3·57 g./cm.^{3} at a depth of 80 km., and thereafter a fairly steady density gradient down to a depth of 2700 km.; the gradient increases sharply in the next 200 km., the density being 5·57 g./cm.^{3} at the base of the mantle. The density inside the central core ranges from 9·74 g./cm.^{3} to 12·00 g./cm.^{3} at a depth of 4980 km. and then rises sharply, reaching a value between 17 and 18 g./cm.^{3} at the centre. Seismic S waves would be transmitted through the inner core at speeds close to 5·0 km./sec. The distributions of the pressure p and certain other properties are shown in tables, and an equation is given connecting p and the incompressibility k for pressures between 0·4 and $3{\cdot}2 \times 10^{12}$ dynes/cm.^{2}. Various uncertainties are discussed, and also the relation of the new model to an earlier model.

* The full text of this paper will be published in *M.N., Geophys. Suppl.*, **6**, No. 1, 1950.

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